The Characteristic Polynomial of a Product

Ralph Howard

This is a note to prove the most notorious of all qualifying exam questions:

Theorem *Let A and B be $n \times n$ matrices over a field $\mathbf{F}$. Then the characteristic polynomials of AB and BA are the same.*

Remark 1 If one of the two matrices, say A, is invertible then $A^{-1}(AB)A = BA$. Thus AB and BA are similar which certainly implies AB and BA have the same characteristic polynomial. However if A and B are both nonsingular then AB and BA do not have to be similar. For example if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

then $AB = B \neq 0$, but $BA = 0$ so that AB and BA are not similar.

Remark 2 I leaned of the following proof from Tom Markham who said it is originally due to Paul Halmos.

PROOF: For any square matrix let C let $\operatorname{char}_C(\lambda) := \det(\lambda I - C)$ be the characteristic polynomial of C. Let r be the rank of A. Then by doing row and column reduction there are invertible $n \times n$ matrices P and Q so that

$$A = P \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Q$$

where I is the $r \times r$ identity matrix. Now express B as

$$B = Q^{-1} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} P^{-1}$$

Where B_{11} is $r \times r$, B_{22} is $(n-r) \times (n-r)$ etc. Then

$$AB = P \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} QQ^{-1} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} P^{-1} = P \begin{bmatrix} B_{11} & B_{12} \\ 0 & 0 \end{bmatrix} P^{-1}$$

so that $\operatorname{char}_{AB}(\lambda) = \lambda^{n-r} \operatorname{char}_{B_{11}}(\lambda)$. Likewise

$$BA = Q^{-1} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} P^{-1} P \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Q = Q^{-1} \begin{bmatrix} B_{11} & 0 \\ B_{21} & 0 \end{bmatrix} Q$$

so that $\operatorname{char}_{BA}(\lambda) = \lambda^{n-r} \operatorname{char}_{B_{11}}(\lambda)$. This completes the proof. $\square$