Grades on the First Exam.

Here is the information on the first test. 34 people took the exam. The high scores were 99, 96, 94, 90, and 90. The low scores were 22, 30, 35, and 36. The average was 71.15 with a standard deviation of 19.68. The median was 73. The break down in the grades is in the table.

Grade	Range	Number	Percent
A	90–100	5	14.71%
B	80–89	10	29.42%
C	70–79	5	14.71%
D	60–69	6	17.65%
F	0–59	8	23.53%

Warning.

This exam was the easiest one of the term. Therefore if you did not do well on it you are in trouble. The last day to drop the class without getting a WF is Thursday October. Judging from what I have seen in past classes, ***anyone who got below 70 on this exam is very likely better off dropping the course now.***

Test 1 Name: Key

Show your work! Answers that do not have a justification will receive no credit. **The questions are worth 10 points each.**

(1) Solve the initial value problem $y''(t) = 4 + 6t$, $y(0) = 1$, $y'(0) = 2$.

$y'(t) = \int (4+6t)\,dt$

$= 4t + 3t^2 + C_1$

$y'(0) = 2 = 4\cdot 0 + 3\cdot 0^2 + C_1$

so $C_1 = 2$

$y' = 4t + 3t^2 + 2$

$y = \int (4t + 3t^2 + 2)\,dt$

$= 2t^2 + t^3 + 2t + C_2$

$y(0) = 0 + C_2 = 1$

so $C_2 = 1$ Thus

$y(t) = \underline{t^3 + 2t^2 + 2t + 1}$

(2) Solve the initial value problem $xy' + 3y = 15x^2$, $y(1) = 5$.

This is a linear equation.

(*) $y' + \frac{3}{x} y = 15x$

Int. Factor $= e^{\int \frac{3}{x}dx} = e^{3\ln|x|} = x^3$

multiple (*) by x^3

$x^3 y' + 3x^2 y = 15x^4$

$(x^3 y)' = 15x^4$

$x^3 y = \int 15x^4\,dx = 3x^5 + C$

$y = 3x^2 + \frac{C}{x^3}$

$y(1) = 3\cdot(1)^2 + \frac{C}{(1)^3} = 5$

$3 + C = 5$

$C = 2$

$y(x) = \underline{3x^2 + \frac{2}{x^3}}$

(3) Find the general solution to $y' = \dfrac{6x^2+1}{4y+3}$

This is separable

Multiply by $4y+3$

$(4y+3)\,dy = (6x^2+1)\,dx$

$\int (4y+3)\,dy = \int (6x^2+1)\,dx$

$2y^2 + 3y = 2x^3 + x + C$

$\underline{2y^2 + 3y = 2x^3 + x + C}$

(4) Find the general solution to $y' = (x+y)^2 - 1$

Do the substitution $v = x+y$

i.e. $y = v - x$

$$y' = v' - 1 = v^2 - 1$$

$$\frac{dv}{dx} = v^2$$

$$v^{-2}\,dv = dx$$

$$\int v^{-2}\,dv = \int dx$$

$$-v^{-1} = x + C$$

$$-(x+y)^{-1} = (x+c)$$

$$(x+y) = -(x+c)^{-1}$$

$$y = -x - (x+c)^{-1}$$

$$\underline{y = -x - \frac{1}{x+C}}$$

(5) Find the general solution to $y' + 2y = 8e^x\sqrt{y}$ $= 8e^x y^{\frac{1}{2}}$

This is Bernoulli with $n = \frac{1}{2}$

$$y = v^{\frac{1}{1-n}} = v^{\frac{1}{1-\frac{1}{2}}} = v^2$$

$$y' = 2vv'$$

$$2vv' + 2v^2 = 8e^x(v^2)^{\frac{1}{2}} = 8e^x v$$

divide by $2v$

$$v' + v = 4e^x$$

int. factor $= e^{\int 1 dx} = e^x$

$$e^x v' + e^x v = 4e^{2x}$$

$$(e^x v)' = 4e^{2x}$$

$$e^x v = \int 4e^{2x}\,dx$$

$$= 2e^{2x} + C$$

$$\sqrt{y} = v = 2e^x + ce^{-x}$$

$$y = (2e^x + ce^{-x})^2$$

$$\underline{y = (2e^x + ce^{-x})^2}$$

(6) Find the general solution to $y' = \dfrac{y}{x} + e^{y/x}$.

This is homogeneous

Let $y = xv$, $v = y/x$

$$y' = xv' + \cancel{v} = \cancel{v} + e^v$$

$$x\frac{dv}{dx} = e^v$$

$$e^{-v}\,dv = \frac{dx}{x}$$

$$-e^{-v} = \ln|x| + C$$

$$-e^{-\frac{y}{x}} = \ln|x| + C$$

$$-\frac{y}{x} = \ln(-\ln|x| + c)$$

$$y = -x\ln(-\ln|x| + c)$$

$$\underline{y = -x\ln(-\ln|x| + c)}$$

(7) Find the general solution to $(2xy^3 + 2x) + (3x^2y^2 + 3y^2)y' = 0$

Rewrite as

$(2xy^3+2x)dx + (3x^2y^2+3y^2)dy = 0$

$x^2y^3 + x^2 + y^3 = C$

$\frac{\partial}{\partial y}(2xy^3) = 6xy^2 = \frac{\partial}{\partial x}(3x^2y^2+3y^2)$

So this is exact.

$\frac{\partial F}{\partial x} = 2xy^3+2x$

$F = \int (2xy^3+2x)dx$

$= x^2y^3 + x^2 + c(y)$

$\frac{\partial F}{\partial y} = 3x^2y^2 + c'(y) = 3x^2y^2+3y^2$

so $c'(y) = 3y^2$

$c(y) = \int 3y^2 dy = y^3 + C$

(8) The starling is not native to the United States, but was introduced to this country when 24 starlings were released into central park in the early 1800's. Five years after their release there where 240 of them. Assuming that the rate of increase of the population of starlings was proportional to the number in the population, how long after the initial release before there were 1,000,000 starlings in the United States?

Number of years = 23.098... years

If $y(t)$ = number of starlings t years after introduction, then $y' = ky$ for some constant k. This has solution

$y(t) = y(0)e^{kt} = 24e^{kt}$

$y(5) = 24e^{5k} = 240$

$e^{5k} = 10$

$k = \frac{1}{5}\ln(10) = .460517\ldots$

$y = 24e^{\frac{1}{5}\ln(10)t}$

Want t so that

$24e^{\frac{1}{5}\ln(10)t} = 1,000,000$

$e^{\frac{1}{5}\ln(10)t} = \frac{1,000,000}{24}$

$\frac{1}{5}\ln(10)t = \frac{1,000,000}{24}$

$t = \frac{5\ln(1,000,000/24)}{\ln(10)}$

$= 23.098$ years

(9) Recall that by Newton's law of cooling an object cools at a rate proportional to the difference between its temperature and the temperature of the surrounding air. A cup of water cooled to 50°F is set in a room that is 75°F. Its temperature 10 minutes latter is 60°F.

(a) Find a formula for the temperature of the cup after t minutes.

If $y(t)$ = temp. t minutes after cup is put in room

Temperature is: $75-25e^{-.05108t}$

$$\frac{dy}{dt} = k(y-75)$$

$$\frac{dy}{y-75} = k\,dt$$

$$\int \frac{dy}{y-75} = \int k\,dt$$

$$\ln(y-75) = kt + C_1$$

$$y-75 = e^{kt+C_1} = C_2 e^{kt}$$

$$y = 75 + C_2 e^{kt}$$

$$y(0) = 75 + C_2 = 50 \text{ so } C_2 = -25$$

$$y(t) = 75 - 25e^{kt}$$

Now

$$y(10) = 75 - 25e^{10k} = 60$$

$$-25e^{10k} = -15$$

$$e^{10k} = \frac{15}{25} = \frac{3}{5} = .6$$

$$10k = \ln(.6)$$

$$k = \frac{\ln(.6)}{10} = -.0518$$

(b) How long does it take for the cup to reach 70°F?

$t =$ 31.5066 min

solve

$$75 - 25e^{-.05108t} = 70$$

$$-25e^{-.05108t} = -5$$

$$e^{-.05108t} = \frac{5}{25} = .2$$

$$t = \frac{\ln(.2)}{-.05108} = 31.5066 \text{ min.}$$

(10) A tank holds 200L of water when it is full. At an initial time the tank contains 100L of pure water. At this time water that has 2Kg per L of salt dissolved in it is pumped in at the rate of 3L/min, and at the same time water is drained out at 2L/min.

(a) What is the volume of water in the tank after t minutes?

$$V' = (\text{rate in}) - (\text{rate out}) = 3 - 2 = 1$$

so $V(t) = 100 + t$

$V(t) =$ $100 + t$

(b) How long before the tank is full?

set $100 + t = 200$ so

$t = 100$

(c) Give a formula for the amount of salt in the tank after t minutes.

$2(t+100) - \dfrac{2{,}000{,}000}{(t+100)^2}$

If $y(t)$ = amount of salt after t minutes.

$$y' = (\text{rate in}) - (\text{rate out})$$

$$= (3)(2) - (2)\left(\frac{y}{t+100}\right)$$

flow rate, concentration, flow rate, concentration

$$y' + \frac{2}{t+100}y = 6$$

$$\text{Int. Factor} = e^{\int \frac{2}{t+100}\,dt} = e^{2\ln(t+100)} = (t+100)^2$$

$$(t+100)^2 y' + 2(t+100)y = 6(t+100)^2$$

$$\left((t+100)^2 y\right)' = 6(t+100)^2$$

$$(t+100)^2 y = \int 6(t+100)^2\,dt = 2(t+100)^3 + C$$

$$y = 2(t+100) + \frac{C}{(t+100)^2}$$

$$y(0) = 2(100) + \frac{C}{(100)^2} = 0$$

$$\text{so} \quad C = -2(100)(100)^2 = -2{,}000{,}000.$$