Math 554 Homework 1

Here are the axioms we have for the order $<$ on the real numbers $\mathbb{R}$ (see the text, Page 2).
(F) For $a, b \in \mathbb{R}$ exactly one of the following holds

$$a < b, \quad a = b, \quad \text{or} \quad b < a.$$

(Trichotomy principle.)
(G) If $a < b$ and $b < c$ then $a < c$ (transitivity).
(H) If $a < b$ then for any c we have $a + c < b + c$. If $c > 0$ then we also have $ac < bc$.

In class we showed

Proposition 1. *If $a < b$ and $c < d$, then $a + c < b + d$.* $\square$

Proposition 2. *If $a, c > 0$ and $a < b$ and $c < d$, then $ac < bd$.*

Problem 1. Prove this.

Problem 2. Show $a > 0$ implies $-a < 0$. *Hint:* By trichotomy one of $-a < 0$, $-a = 0$, or $-a > 0$ holds. Show that $-a = 0$ and $-a > 0$ each lead to a contradiction, which only leaves $-a < 0$.

Problem 3. Show $a < 0$ implies $-a > 0$.

Combining the last two problems gives

Proposition 3. *For $a \in \mathbb{R}$ we have $a > 0$ if and only if $-a < 0$.* $\square$

We use the standard notations $a \leq b$ to mean $a < b$ or $a = b$ and $a > b$ to mean $b < a$ etc.

Proposition 4. *For $a, b \in \mathbb{R}$ show*
(a) *$a > 0$ and $b < 0$ implies $ab < 0$,*
(b) *$a < 0$ and $b > 0$ implies $ab < 0$, and*
(c) *$a < 0$ and $b < 0$ implies $ab > 0$.*

Problem 4. Prove this.

Definition 5. If $a \in \mathbb{R}$ the ***absolute value*** of a is

$$|a| := \begin{cases} a, & a \geq 0; \\ -a, & a < 0. \end{cases}$$

Geometrically $|a|$ is the distance of a from the origin.

Proposition 6. *For $a \in \mathbb{R}$*
(a) $|a| \geq 0$,
(b) $|-a| = |a|$,
(c) $a \leq |a|$, *and*
(d) $|a|^2 = a^2$.

Problem 5. Prove this.

Proposition 7. *If* $a, b \in \mathbb{R}$ *then* $|ab| = |a||b|$.

Problem 6. Prove this.

Proposition 8. *For* $a, b \in \mathbb{R}$

$$|a| \le |b| \quad \textit{if and only if} \quad a^2 \le b^2.$$

Problem 7. Prove this.

Theorem 9 (Triangle inequality)**.** *If* $a, b \in \mathbb{R}$ *then*

$$|a + b| \le |a| + |b|.$$

Problem 8. Prove this. *Hint:* In light of Proposition 8 it is enough to show $|a+b|^2 \le (|a| + |b|)^2$. But $|a+b|^2 = (a+b)^2 = a^2 + 2ab + b^2$ and $(|a| + |b|)^2 = |a|^2 + 2|a||b| + |b|^2 = a^2 + 2|a||b| + b^2$.