Math 554 Review for Test 3.

The basic topic we have covered since the last test is differentiation. Here are some basic things you should know about derivatives.

1. The definition of the derivative of a function f at a point x_0, knowing how to apply it in simple cases and its basic consequences.

(a) Being able to show a function is differentiable at a point from the definition. This comes up when you have a function that is not easily seen to be differentiable from our results about the differentiability of polynomials, rational functions, and the like. Here is an example of such a problem similar to one that caused trouble on the homework. *Example:* Show the function

$$f(x) = \begin{cases} x^2 \sin(1/x^2), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

is differentiable at 0. *Solution:*

$$\begin{aligned} f'(0) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \to 0} \frac{x^2 \sin(1/x^2)}{x} \\ &= \lim_{x \to 0} x \sin(1/x^2) \end{aligned}$$

But $-|x| \leq x \sin(1/x^2) \leq |x|$ and $\lim_{x \to 0} -|x| = \lim_{x \to 0} |x| = 0$ and therefore by the Squeeze Lemma $\lim_{x \to 0} x \sin(1/x^2) = 0$. Thus $f'(0)$ exists and has the value $f'(0) = 0$.

(b) Here is another example. If f is defined on $\mathbb{R}$ and satisfies $|f(x) - f(y)| \leq 5|x - y|^2$, then show f is differentiable and find its derivative. *Solution:* For any $x_0 \in \mathbb{R}$ we have

$$0 \leq \left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq \frac{5|x - x_0|^2}{|x - x_0|} = 5|x - x_0|$$

and as $\lim_{x \to x_0} 5|x - x_0| = 0$ another application of the Squeeze Lemma shows

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = 0$$

for all x_0. This also shows that f is constant.

(c) You should be able to prove the basic derivatives formulas and be able to state them precisely. Here is an example of such a statement.

Proposition 1. *Let f and g be defined in a neighborhood of x_0 and assume that $f'(x_0)$ and $g'(x_0)$ exist. Then the function h defined by $h(x) = f(x)g(x)$ is differentiable at x_0 and $h'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.*

It is important that you state that the product is differentiable.

2. The equivalent definition of differentiability in terms of linear approximation. Here is the result.

Theorem 2. *Let f be defined on a neighborhood of x_0. Then the following are equivalent.*

(i) *f is differentiable at x_0.*

(ii) *There is a constant m and a function $E(x)$ such that*

$$f(x) - f(x_0) = m(x - x_0) + E(x)(x - x_0)$$

and

$$\lim_{x \to x_0} E(x) = 0.$$

If these hold, them $m = f'(x_0)$.

You should know the statement of this and look at the proof. Certainly the proof that (ii) $\implies$ (i) would be a fair question. And the proof of (i) $\implies$ (i) is not that hard.

To date the only application of this theorem we have is the proof of the chain rule.

Theorem 3 (Chain Rule)**.** *Let g be differentiable at x_0 and f differentiable at $g(x_0)$. The the composition $h = f \circ g$ is differentiable at x_0 and*

$$(f \circ g)'(x_0) = f'(g(x_0))g'(x_0).$$

Proof. Theorem 3 and the differentiability assumptions about f and g imply that there are functions E_f and E_g with

$$g(x) - g(x_0) = g'(x_0)(x - x_0) + E_g(x)(x - x_0)(x - x_0) \tag{1}$$

$$f(y) - f(g(x_0)) = f'(g'(x_0))(y - g(x_0)) + E_f(y)(y - g(y_0)) \tag{2}$$

where

$$\lim_{x \to x_0} E_g(x) = 0, \qquad \lim_{y \to g(x_0)} E_f(y) = 0. \tag{3}$$

Letting $y = g(x)$ in (2) and then using (1) gives

$$\begin{aligned} f(g(x)) - f(g(x_0)) &= f'(g'(x_0))(g(x) - g(x_0)) + E_f(g(x))(g(x) - g(x_0)) \\ &= f'(g'(x_0))\Big(g'(x_0)(x - x_0) + E_g(x)(x - x_0)\Big) \\ &\quad + E_f(g(x))\Big(g'(x_0)(x - x_0) + E_g(x)(x - x_0)\Big) \\ &= f'(g'(x_0))g'(x_0)(x - x_0) + E(x)(x - x_0) \end{aligned}$$

where

$$E(x) = f'(g'(x_0))E_g(x) + E_f(g(x))g'(x_0) + E_f(g(x))E_g(x)$$

and, using the limits (3), is not hard to see

$$\lim_{x \to x_0} E(x) = 0.$$

Therefore, by Theorem 3, the function $f \circ g$ is differentiable at x_0 with derivative $f'(g(x_0))g'(x_0)$. □

Being able to set this up would be a fair test question.

3. Anther major topic was the ***Extreme Value Theorem*** (Theorem 3 on the *Homework due Wednesday, November 14* from the class web page, although I did not give it a name there), using this to prove Rolle's Theorem, and then using Rolle's Theorem to prove the mean value theorem and the generalized mean value theorem. Here I don't have much to add to what was said on the homework just referenced. You will certainly have to use the mean value theorem to prove something like

$$|(a^3 + 2a) - (b^3 + 2b)| \geq 2|b - a|.$$

Or that if $|f'(x)| \leq 5$ then $|f(x) - f(y)| \leq 5|x - y|$. (Hopefully I can come up with some problems of this type that are bit more fun than these two examples.)

4. We then used the generalized mean value theorem to prove l'hôpital's rule. And we generalized Rolle's and used that to prove Taylor's theorem with Lagrange's form of the remainder. And again I don't have anything to add to what is on the relevant homework assignments. Of course be able to state all these results.

5. We will/may cover one more topic between now and the test. That is the derivative of inverse functions. You will get notes on this after the break.

6. As usual there will be surprise mystery questions.