Math 554, Test 2 Name: Answer Key.

1. (a) Define (E, d) is a ***metric space***.

Answer: E is a nonempty set and $d\colon E \times E \to \mathbf{R}$ is a function such that for all $x, y, z \in E$

- $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$.
- $d(x, y) = d(y, x)$ (Symmetry.)
- $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle inequality.)

(b) Define $B(p, r)$ is the ***open ball*** of radius r centered at p.

Answer: $B(p, r) := \{x \in E : d(x, p) < r\}$.

Or if you prefer English $B(p, r)$ is the set of points in E that are at a distance less that r from p.

(c) Define V is an ***open set*** in E.

Answer: The set V is open iff for all $p \in V$ there is an $r > 0$ such that $B(p, r) \subseteq V$.

Or in English. A set V is open iff for any point, p, of V there is an open ball centered at p and contained in V.

(d) Define S is a ***closed set*** in E.

Answer: The set S is closed iff its complement, $\mathcal{C}S$, is open.

(e) Define $\langle p_n \rangle_{n=1}^{\infty}$ is a ***Cauchy sequence*** in the metric space E.

Answer: The sequence $\langle p_n \rangle_{n=1}^{\infty}$ in the metric space E is a Cauchy sequence iff for any $\varepsilon > 0$ there an N such that

$$m, n > N \qquad \Longrightarrow \qquad d(p_m, p_n) < \varepsilon.$$

(f) Define what it means for the metric space E to be ***complete***.

Answer: The metric space E to be complete iff every Cauchy sequence in E converges to a point in E.

(g) Define what it means for p to be a ***limit point*** of the set S.

Answer: The point $p \in E$ is a limit point of S iff there is a sequence $\langle p_k \rangle_{k=1}^{\infty}$ with

$$\lim_{k \to \infty} p_k = p$$

and $p_k \in S$ for all k.

2. If U and V are open sets in the metric space E prove $U \cap V$ is also open in E.

Answer: Let $p \in U \cap V$. As U is open there is an $r_1 > 0$ such that $B(p, r_1) \subseteq U$. As V is open there is an $r_2 > 0$ such that $B(p, r_2) \subseteq V$. Let $r = \min\{r_1, r_2\}$. Then $B(p, r) \subseteq B(p, r_1) \subseteq U$ and $B(p, r) \subseteq B(p, r_2) \subseteq V$ which implies that $B(p, r) \subseteq U \cap V$. Thus $U \cap V$ contains an open ball about its point p. As p was any point of $U \cap V$ this implies that $U \cap V$ is open.

3. (a) If E is a metric space and $\langle p_n \rangle_{n=1}^{\infty}$ is sequence in E define $\lim_{n\to\infty} p_n = p$.

Answer: For all $\varepsilon > 0$ there a N such that

$$n > N \qquad \implies \qquad d(p_n, p) < \varepsilon.$$

(b) If $\langle a_n \rangle_{n=1}^{\infty}$ and $\langle b_n \rangle_{n=1}^{\infty}$ are sequences in $\mathbf{R}$ with

$$\lim_{n\to\infty} a_n = a \qquad \text{and} \qquad \lim_{n\to\infty} b_n = b$$

then give an ε and N proof that

$$\lim_{n\to\infty} (3a_n + 2b_n) = 3a + 2b.$$

Answer: As $\lim_{n\to\infty} a_n = a$ there an N_1 such that

$$n > N_1 \qquad \implies \qquad |a_n - a| < \frac{\varepsilon}{6}.$$

Likewise as $\lim_{n\to\infty} b_n = b$ there an N_2 such that

$$n > N_2 \qquad \implies \qquad |b_n - b| < \frac{\varepsilon}{4}.$$

Let $N = \max\{N_1, N_2\}$. Then for $n > N$ we have

$$\begin{aligned} |(3a_n - 2b_n) - (3a + 2b)| &= |(3a_n - a) - 2(b_n - b)| \\ &\le 3|a_n - a| + 2|b_n - b| \\ &< 3\frac{\varepsilon}{6} + 2\frac{\varepsilon}{4} \\ &= \varepsilon. \end{aligned}$$

Thus $\lim_{n\to\infty}(3a_n + 2b_n) = 3a + 2b$.

4. (a) Give and example of metric space E and a Cauchy sequence in E that is does not converge. (Just give the example, you do not have to prove it works.)

Answer: An easy example is $E = (0, 1)$ and letting the sequence be $a_n = 1/n$.

(b) Give an example of subset of $\mathbf{R}$ that is neither open or closed. (Just give the example, you do not have to prove it works.)

Answer: The half open interval $[0, 1)$ does the trick.

5. Prove that if the sequence $\langle p_n \rangle_{n=1}^{\infty}$ is convergent then it is Cauchy.

Answer: Let $\langle p_n \rangle_{n=1}^{\infty}$ be a convergent sequence in the metric space E. This means there is a point $p \in E$ such that $\lim_{n\to\infty} p_n = p$. Thus there is a N with

$$n > N \qquad \Longrightarrow \qquad d(p_m, p) < \frac{\varepsilon}{2}.$$

Then if $m, n > N$ we can use the triangle inequality to get

$$d(p_m, p_n) \le d(p_n, p) + d(p, p_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which shows that the sequence is Cauchy.

6. (a) If E is a metric space and S a subset of E, then define what it means for $\mathcal{V}$ to be an ***open cover*** of S.

Answer: $\mathcal{V}$ is an open cover of S iff every $V \in mathcal V$ is an open subset of E and for each $x \in S$ there is a $V \in \mathcal{V}$ with $x \in V$.

Here is a slightly different wordiing that is correct: $\mathcal{V}$ is an open cover of S iff eacy $V \in mathcal V$ is an open subset of E and $S \subseteq \bigcup_{V\in\mathcal{V}} V$.

(b) Define what it means for the subset S of the metric space E to be ***compact***.
The subset S of E is compact iff every open cover S has a finite subset that still covers S.

(c) Define what it means for the subset of S of the metric space to be ***bounded***.

Answer: The subset S of E is bounded iff there is ball $B(p, r)$ with $S \subseteq B(p, r)$.

(d) Prove that a compact subset of a metric space is bounded.

Answer: Let S be compact in E. Choose a point $p \in E$ and set

$$\mathcal{V} := \{B(p, r) : r > 0\}$$

Then every element of $\mathcal{V}$ is an open ball and thus an open set. If $x \in S$ then let r be such that $r > d(p, x)$. Then $x \in B(p, r) \in \mathcal{V}$. Thus $\mathcal{V}$ is an open cover of S.

As S is compact there there is a finite subset $\{B(p, r_1), \ldots, B(p, r_n)\} \subseteq \mathcal{V}$ with

$$S \subseteq \bigcup_{k=1}^{n} B(p, r_k).$$

Let $r = \max\{r_1, r_2, \ldots, r_n\}$. Then $\bigcup_{k=1}^{n} B(p, r_k) = B(p, r)$. Thus

$$S \subseteq \bigcup_{k=1}^{n} B(p, r_k) = B(p, r)$$

which shows that S is bounded.