Math 574 **Test 3** Name:__________________

You are to use your own calculator, no sharing.
Show your work to get credit.

1. Find the sums of the following series.
(a) $a + ar + ar^2 + ar^3 + \cdots$ where $|r| < 1$.
Solution:

$$\text{Sum} = \frac{a}{1-r}$$

□

(b) $\displaystyle\sum_{k=1}^{\infty} x^{2k}$ where $|x| < 1$.
Solution:

$$\text{Sum} = \frac{x^2}{-1x^2}$$

□

2. (Recall that Newton's binomial theorem is that for $|x| < 1$ and any real number α

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n.$$

(a) Give a formula for $\displaystyle\binom{\alpha}{n}$.
Solution: It is given by

$$\binom{\alpha}{n} = \frac{n^{\underline{r}}}{r!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}.$$

(b) From basic algebra we know

$$(1+x)^{\alpha}(1+x)^{-\alpha} = 1.$$

Use this to derive a find a simple expression for the sum

$$\sum_{i+j=n} \binom{\alpha}{i}\binom{-\alpha}{j}$$

Solution:

$$\begin{aligned} 1 &= (1+x)^{\alpha}(1+x)^{-\alpha} \\ &= \sum_{n=0}^{\infty} \left(\sum_{i+j=n} \binom{\alpha}{i}\binom{-\alpha}{j} \right) x^n \end{aligned}$$

Comparing the coefficients we find

$$\sum_{i+j=n} \binom{\alpha}{i}\binom{-\alpha}{j} = \begin{cases} 1, & n = 0; \\ 0, & n > 0. \end{cases}$$

Here we have used that $1 = 1 + 0x + 0x^2 + 0x^3 + \cdots$. □

3. There is a country where stamps come in three denominations. 1¢, 3¢ and 5¢. Let S_n be the number of ways to put n¢ postage on a letter. What is the generating function for $\sum_{n=0}^{n} S_n x^n$? (5 points extra credit if you give a closed form for the answer.)

Solution: The generating function is

$$f(x) = (1 + x + x^2 + x^3 + \cdots)(1 + x^3 + x^6 + x^9 + \cdots)(1 + x^5 + x^{10} + x^{15} + \cdots)$$
$$= \frac{1}{(1-x)(1-x^3)(1-x^5)}.$$

□

4. Explain how to find the number of solutions to $a + b + c + d = 30$ where a is a nonnegative integer, b is a positive integer, c is an odd number and d is nonnegative and dividable by 3. Your answer should be something like: This number is the coefficient of $x^{\text{you fill this in}}$ in the series (and here you give the series).

Solution: The generating function for this is

$$f(x) = (1 + x + x^2 + \cdots)(x + x^2 + x^3 + \cdots)(x + x^3 + x^5 + x^7 + \cdots)(1 + x^3 + x^6 + x^9 + \cdots)$$

and the number we are looking for is the coefficient of x^{30} in this series. (In case your really wanted to know this coefficient is 865.) □

5. Let A and B be two subsets of the nonnegative integers and let

$$f_A(x) = \sum_{a\in A} x^n \qquad \text{and} \qquad f_B(x) = \sum_{b\in B} x^b.$$

Explain why the coefficient of x^n in the product

$$f_A(x)f_B(x)$$

is the number of solutions to $a + b = n$ with $a \in A$ and $b \in B$.

Solution: One way to thing of this is that

$$f_A(x)f_B(x) = \sum_{a\in A,\, b\in B} x^a x^b = \sum_{a\in A,\, b\in B} x^{a+b}.$$

We now combine the terms that give x^n to see

$$f_A(x)f_B(x) = \sum_{n=0}^{]} infty \left(\sum_{a\in A, b\in B, a+b=n} 1 \right) x^n = \sum_{n=0}^{\infty} S_n x^n$$

where

$$S_n = \text{number of solutions to } a + b = n \text{ with } a \in A \text{ and } b \in B.$$

□

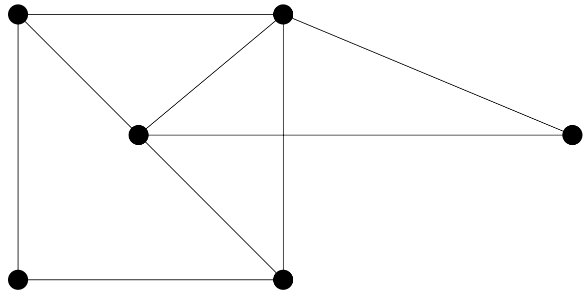
6. (a) Define what it means for a graph to have an ***Eulerian circuit***.

Solution: It is a cycle that pass though each edge exactly one time. □

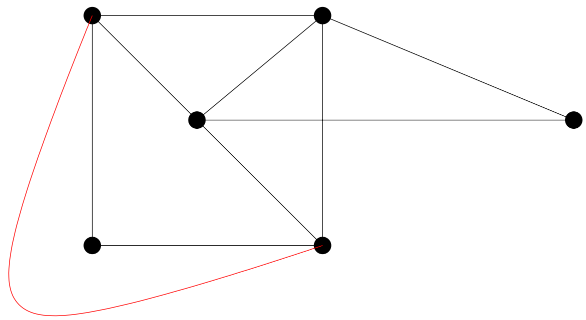
(b) State our theorem about when a connected simple graph has an Eulerian circuit.

Solution: A simple connected graph is Eulerian if and only if the degrees of all the vertices are even. □

(c) Is the following graph Eulerian? If it is not add the smallest possible number of edges that will make it Eulerian.



Solution: It is not Eulerian as two of the vertices have degree 3 and 3 is odd. So we have to add at least one edge to make it Eulerian. If we add an edge that connects the two vertices of degree 3, then all degrees will be even and so the resulting graph will be Eulerian. In the following picture the new edge is in red.



□

7. (a) State the result about the sum of the degrees of all the vertices of the a graph with no loops.

Solution: In a graph with no loops the sum of the degrees of the vertices is twice the number of edges. In symbols

$$\sum_{v \in V} \deg(v) = 2\#(E).$$

□

(b) Use the theorem of (a) to explain why in a graph with no looks the sum of the degrees of all the vertices is even.

Solution: The sum of the degrees of the vertices is $2\#(E)$ and this is an even number. □

(c) Is a possible to have a group of 23 people where each of them has exactly 5 friends in the group? (Assume that friendship is symmetric, that is if A is friends with B, then B is friends with A.) Explain your answer.

Solution: In is impossible. To see this assume that everyone did have exactly 5 friends. Then consider the graph with vertices the people in the group and where there is an edge between two people if there are friend. As each person as exactly 5 friends the degree of each vertex is 5. Thus

$$\sum_{v \in V} \deg(v) = 5\#(V) = 5 \cdot 23 = 115$$

which is odd. But this is impossible by part (b). □