Mathematics 554 Homework.

The big new idea in the class is the least upper bound axiom. This is that any nonempty set $S \subseteq \mathbb{R}$ which has an upper bound has a least upper bound, denoted $\sup(S)$ and is also called the ***supremum*** of S. Here are some problems on using the least upper bound axiom.

Proposition 1 (Generalization of Archimedes Axiom)**.** *Let $A \subseteq \mathbb{R}$ be a nonempty subset of $\mathbb{R}$ such that for some positive number δ, if $a \in A$, then so is $a + \delta$. Then for every $x \in \mathbb{R}$ there is an $a \in A$ with $a > x$.*

Problem 1. Prove this. *Hint:* Toward a contradiction assume this is false, then there is an $x \in \mathbb{R}$ with $a \leq x$ for all $a \in A$. That is x is an upper bound for A. Therefore by the least upper bound axiom A has a supremum. Let $b = \sup(A)$. Then for every $a \in A$ we also have $a + \delta \in A$ and thus $a + \delta \leq b$ as b is an upper bound for A. Use this and the definition of supremum to derive a contradiction. $\square$

Proposition 2 (Existance of greatest integer)**.** *For every $x \in \mathbb{R}$ there is a unique integer n such that*

$$n \leq x < n + 1.$$

*This n is the **greatest integer** in x, also called the **floor** of x.*

Problem 2. Prove this. This is one of those propositions that is just plain irritating to prove. Here is an outline of one proof. To start let $x \in \mathbb{R}$ and let

$$S = \{m \in \mathbb{Z} : m \leq x\}$$

where $\mathbb{Z}$ is the set of integers.

(a) Show that $S \neq \varnothing$. *Hint:* By Archimedes' Axiom that is a natural number k with $-x < k$. Show $-k \in S$.

(b) As S is bounded above (why?), it has a supremum, let

$$\alpha = \sup(S).$$

Explain why there is an integer $n \in S$ with

$$\alpha - 1 < n \leq \alpha$$

and why

$$\alpha \leq x.$$

(c) Thus

$$\alpha < n + 1 \leq \alpha + 1 \leq x + 1$$

Use this to show $n \leq x$.

(d) Use $\alpha < n + 1$ to show $n + 1 \notin S$ and say why this implies $x < n + 1$.

(e) Use parts (c) and (d) to show

$$n \leq x < n + 1.$$

This completes the proof of the existence of n.

(f) To show uniqueness, assume that m is an integer with

$$m \leq x < m + 1.$$

Manipulate these inequalities and the inequalities $n \leq x < n + 1$ to get

$$-1 < n - m < 1$$

and explain why this implies $m = n$. *Hint:* There is only on integers between -1 and 1. What is it?