

## Mathematics 554 Homework.

**Definition 1.** Let  $E \subseteq \mathbb{R}$ . Then a function  $f: E \rightarrow \mathbb{R}$  is **Lipschitz** if and only if there is a constant  $M \geq 0$  so that

$$|f(x_2) - f(x_1)| \leq M|x_2 - x_1|$$

for all  $x_1, x_2 \in E$ . The constant  $M$  is a **Lipschitz constant** for  $f$ .  $\square$

Many of the functions you have met in calculus are Lipschitz when  $E = [a, b]$  is a bounded interval.

**Problem 1.** Do Problem 2.46 on Page 44 of the notes.

**Proposition 2.** On a bounded interval  $[a, b]$  the function  $f(x) = x^n$  is Lipschitz for any positive integer  $n$ .

**Problem 2.** Prove this. *Hint:* First note for  $x \in [a, b]$  that

$$|x| \leq \max(|a|, |b|).$$

(You do not have to prove this, we have done enough exercises on absolute values.) To simplify notation let  $C = \max(|a|, |b|)$ . Then we have  $|x| \leq C$  for all  $x \in [a, b]$ . Here I will do the proof for  $n = 5$  and let you do the proof in the general case. What gets us started is one of the factoring formulas we reviewed at the beginning of the term:

$$\begin{aligned} |f(x_2) - f(x_1)| &= |x_2^5 - x_1^5| \\ &= |x_2 - x_1| (x_2^4 + x_2^3x_1 + x_2^2x_1^2 + x_2x_1^3 + x_1^4) \\ &\leq |x_2 - x_1| (|x_2|^4 + |x_2|^3|x_1| + |x_2|^2|x_1|^2 + |x_2||x_1|^3 + |x_1|^4) \\ &\leq (x_2 - x_1) (C^4 + C^4 + C^4 + C^4 + C^4) \\ &= 5C^4|x_2 - x_1| \\ &= M|x_2 - x_1|. \end{aligned}$$

Where  $M := 5C^4$ . Thus  $f$  is Lipschitz on  $[a, b]$ .  $\square$

**Problem 3.** This problem shows that in Proposition 2 it is important the interval is bounded. Show that  $f(x) = x^2$  is not Lipschitz on the interval  $[0, \infty)$ . *Hint:* Toward a contradiction assume that there is a constant  $M$  with

$$|f(x_2) - f(x_1)| = |x_2^2 - x_1^2| \leq M|x_2 - x_1|.$$

Factoring gives

$$|x_2 - x_1||x_2 + x_1| \leq M|x_2 - x_1|.$$

When  $x_1 \neq x_2$  we can cancel to get

$$|x_2 + x_1| \leq M$$

for all  $x_1, x_2 \in [0, \infty)$  with  $x_1 \neq x_2$ . Now make a choice of  $x_1, x_2$  that leads to a contradiction.  $\square$

**Proposition 3.** Let  $E \subseteq \mathbb{R}$  and  $f_1, f_2: E \rightarrow \mathbb{R}$  be Lipschitz. Then for any constants  $c_1, c_2$  the function  $f = c_1f_1 + c_2f_2$  is also Lipschitz.

*Proof.* By the definition of Lipschitz there are constants  $M_1$  and  $M_2$  with

$$|f_1(x_2) - f_1(x_1)| \leq M_1|x_2 - x_1|, \quad |f_2(x_2) - f_2(x_1)| \leq M_2|x_2 - x_1|.$$

Then

$$\begin{aligned} |f(x_2) - f(x_1)| &= |(c_1f_1(x_2) + c_2f_2(x_2)) - (c_1f_1(x_1) + c_2f_2(x_1))| \\ &= |c_1(f_1(x_2) - f_1(x_1)) + c_2(f_2(x_2) - f_2(x_1))| \\ &\leq |c_1||f_1(x_2) - f_1(x_1)| + |c_2||f_2(x_2) - f_2(x_1)| \\ &\leq |c_1|M_1|x_2 - x_1| + |c_2|M_2|x_2 - x_1| \\ &= (|c_1|M_1 + |c_2|M_2)|x_2 - x_1| \\ &= M|x_2 - x_1| \end{aligned}$$

where  $M = |c_1|M_1 + |c_2|M_2$ . Thus  $f$  is Lipschitz.  $\square$

**Theorem 4.** Let  $p(x) = c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$ . Then  $p$  is Lipschitz on any bounded interval.

*Proof.* A straightforward induction that I am not going to inflict on you (at least not this week).  $\square$

**Theorem 5** (Intermediate Value Theorem for polynomials). Let  $p$  be a polynomial and  $a < b$ . Assume that  $p(a)$  and  $p(b)$  have opposite signs (one way to say this is that  $f(a)f(b) < 0$ ). Then  $p$  has a root in  $(a, b)$ , That is there is a  $\xi \in (a, b)$  so that  $p(\xi) = 0$ .

*Proof.* Now that we know that polynomials are Lipschitz on bounded intervals, this follows from the Lipschitz intermediate Value Theorem on the interval  $[a, b]$ .  $\square$

**Proposition 6.** If  $n$  is a positive integer, then every positive real number has a positive  $n$ -th root. That for all  $c > 0$  there is a positive solution to the equation  $x^n - c$ .

**Problem 4.** Prove this. *Hint:* Let  $p(x) = x^n - c$ . Then  $p(0) = -c < 0$ . Now find a positive  $b$  with  $p(b) > 0$ .  $\square$

**Problem 5.** Let  $p(x) = x^3 + ax^2 + bx + c$ . Prove that  $p(x)$  has at least one real root. *Hint:* The idea is to find  $\alpha$  and  $\beta$  so that  $p(\alpha) < 0$  and  $p(\beta) > 0$ . Then by the intermediate value theorem there is a  $\xi$  between  $\alpha$  and  $\beta$  with  $p(\xi) = 0$ . Write  $p(x)$  as

$$p(x) = x^3 \left( 1 + \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} \right).$$

If  $|x| \leq 1$  then

$$\frac{1}{|x|^3} \leq \frac{1}{|x|^2} \leq \frac{1}{|x|}$$

(you do not have to prove this). Therefore if  $|x| \geq 1$

$$\begin{aligned} \left| \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} \right| &\leq \frac{|a|}{|x|} + \frac{|b|}{|x|^2} + \frac{|c|}{|x|^2} \\ &\leq \frac{|a| + |b| + |c|}{|x|} \end{aligned}$$

To simplify notation let  $S = |a| + |b| + |c|$ , then this becomes

$$\left| \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} \right| \leq \frac{S}{|x|}$$

Use this to show

$$|x| \geq \max(1, 2S) \implies \left| \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} \right| \leq \frac{1}{2}$$

and therefore (give reason)

$$|x| \geq \max(1, 2S) \implies 1 + \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} \geq \frac{1}{2} > 0.$$

To finish let  $\alpha = -2S$  and  $\beta = 2S$  and show  $f(\alpha) < 0$  and  $f(\beta) > 0$ .  $\square$