Mathematics 554 Homework.

Here are few problems to practice working with the definitions of open and closed sets.

Problem 1. Let (E,d) be a metric space and $A \subseteq E$. Let $\overline{A}$ be the set of all points $p \in E$ so that for all $r > 0$ we have $B(p,r) \cap A \neq \varnothing$. Show that $\overline{A}$ is closed. *Hint:* Show the compliment of $\overline{A}$ is open. If q is in the compliment, the writing what this means should come close to finishing the proof. $\square$

Problem 2. Let (E,d) be a metric space. Let $S \subseteq E$ with the property that if $s_1, s_2 \in S$ with $s_1 \neq s_2$, then $d(s_1,s_2) \geq 1$. Show S is closed. *Hint:* First show that any ball $B(p,1/2)$ can contain at most one point of S (use the triangle inequality to show that if $a,b \in B(p,1/2)$, the $d(a,b) < 1$ and explain why this implies $B(p,1/2)$ can contain at most one point of S). Let $U = E \setminus S$ be the compliment of S in E and let $p \in U$, that is $p \notin S$. We need to find an $r > 0$ so that $B(p,r) \cap S = \varnothing$.

Case 1. $B(p,1/2) \cap S = \varnothing$. Then $r = 1/2$ works.

Case 2. $B(p,1/2) \cap S \neq \varnothing$. Then by what we have just shown, $B(p,1/2)$ contains exactly one point of S, call it s. Let $r = d(p,s)$ and explain why $B(p,r) \cap S = \varnothing$. $\square$

Problem 3. In the plane $\mathbb{R}^2$, show the half plane $H = \{(x,y) : y > 0\}$ is open. $\square$

Problem 4. Let (E,d) be a metric space and $p,p \in E$ with $p \neq q$. Show that $U := \{x \in E : d(p,x) < d(q,x)$ is open. $\square$

Problem 5. In $\mathbb{R}$ for the following sets say if they are open, closed, or neither. Prove your answer is correct.

(a) The set, $\mathbb{Q}$, of rational numbers.
(b) The set $\{1/n : n = 1,2,3,\ldots\}$.
(c) The set $\{0\} \cup \{1/n : n = 1,2,3,\ldots\}$. $\square$

Problem 6. Let E be a metric space. Then a subset, $S \subseteq E$ is ***bounded*** if and only if there is a ball $B(a,r)$ with $S \subseteq B(a,r)$. Let $\langle p_n \rangle_{n=1}^{\infty}$ be a convergent sequence. That is there is $p \in E$ so that $\lim_{n\to\infty} p_n = p$. Show the set $\{p_n : n = 1,2,3\ldots\}$ is bounded.

In *Notes on Analysis* Read the first part of Section 3.1 Pages 55–58. I have done some rewriting in the notes to you should look at the current version to make sure the pages and problem numbers agree with the problems I am assigning. o problems 3.18–3.21 and 3.23–3.26. (The one left out, 3.22, is just anther induction and I did not want to make you do anther one of those.)