Mathematics 554 Homework.

The following are questions such as will be on the quiz on Friday.

Problem 1. State what it means for b to be any ***upper bound*** for the nonempty set $S \subseteq \mathbb{R}$.

Solution. The number b is an upper bound for S if and only if $x \le b$ for all $x \in S$. □

Problem 2. State what it means for b to be a ***least upper bound*** for a set S.

Solution. b is a least upper bound for $S \neq \varnothing$ if and only if b is an upper bound for S and if c is any upper bound for S, then $b \le c$. □

Problem 3. State what it means for b to be a ***maximum*** of S.

Solution. b is a maximum of S if and only if $b \in S$ and for all $x \in S$ we have $x \le b$. □

Problem 4. Given an example of a set $S \subseteq \mathbb{R}$ that has no maximum but with $\sup(S) = 1$.

Solution. An easy example is $S = (0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$. □

Problem 5. State the ***least upper bound axiom***.

Solution. Every nonempty subset of $\mathbb{R}$ with an upper bound has a least upper bound. □

If $S \subseteq \mathbb{R}$, then we recall that a function $f\colon S \to \mathbb{R}$ is ***Lipschitz*** if and only if there is a constant M such that

$$|f(x) - f(y)| \le M|x-y|$$

for all $x, y \in S$.

Problem 6. Show that the function $f(x) = x^3 - 2x + 5$ is Lipschitz on the interval $[-10, 10]$.

Solution. Let $x, y \in [-10, 10]$. Then $|x|, |y| \le 10$. Therefore

$$\begin{aligned}
|f(x) - f(y)| &= |(x^3 - 2x + 5) - (y^3 - 2y + 5)| \\
&= |(x^3 - y^3) - 2(x-y)| \\
&= |(x-y)|\big|(x^2 + xy + y^2) - 2\big| && \text{(factoring)} \\
&\le |x-y|\big(|x|^2 + |x||y| + |y|^2 + 2|\big) && \text{(triangle inequality)} \\
&\le |x-y|\big(10^2 + (10)(10) + (10)^2 + 2\big) && \text{(as } |x|, |y| \le 10\text{)} \\
&= 302|x-y| \\
&= M|x-y|
\end{aligned}$$

with $M = 302$. So f is Lipschitz on $[-10, 10]$. □