Mathematics 554 Homework.

In *Notes on Analysis* read pages 51–54 about open and closed sets in metric spaces. Do problems 3.3–3.13.

Here is some review about unions and intersections. Let E be a set and $\mathcal{U}$ a collection of subsets of E. That is if $U \in \mathcal{U}$, then $U \subseteq E$.

Example 1. Let $E = \mathbb{R}$, then

$$\mathcal{U} = \{(-r, r) : r > 0\}$$

the collection of open intervals $(-r, r)$ in $\mathbb{R}$. $\square$

Example 2. Let E be a metric space and $p \in E$. An example that will come up several times is

$$\mathcal{U} = \{B(p, r) : B(p, r) : r > 0\}.$$

This is the set of all open balls with center p in E. $\square$

Example 3. Let E be a metric space and $r > 0$. Then

$$\mathcal{U} = \{B(p, r) : p \in E\}$$

is the collection of all open balls in E with radius r. $\square$

Definition 4. Let E be a set and $\mathcal{U}$ a collection of subsets of E. Then the ***union*** of $\mathcal{U}$ is

$$\bigcup \mathcal{U} = \{x \in E : \ x \in U \text{ for some } U \in \mathcal{U}\}.$$

That is the union of $\mathcal{U}$ is the set of points of E that are in at least one $U \in \mathcal{U}$. $\square$

Definition 5. Let E be a set and $\mathcal{U}$ a collection of subsets of E. Then the ***intersection*** of $\mathcal{U}$ is

$$\bigcap \mathcal{U} = \{x \in E : \ x \in U \text{ for all } U \in \mathcal{U}\}.$$

That is the intersection of $\mathcal{U}$ is the set of points of E that are in all $U \in \mathcal{U}$. $\square$

Often a slightly different notation is used. Let I be some index[1] set and for each $i \in I$ let $U_i \subseteq E$. Then

$$\bigcup_{i \in I} U_i = \bigcup \{U_i : i \in I\} = \{x \in E : x \in U_i \text{ for at least one } i \in I\}$$

and

$$\bigcap_{i \in I} U_i = \bigcap \{U_i : i \in I\} = \{x \in E : x \in U_i \text{ for all } i \in I\}.$$

Here are a few problems to practice working with these ideas.

[1]Here the term "index" does not have any precise meaning. It is just used to name (that is index) the sets U_i.

Problem 1. Let E be a metric space and $p \in E$. Show

$$\bigcup_{r>0} B(p,r) = E.$$

Hint: Do not try to make this hard. If $x \in E$ then note $r = d(p,x) + 1$ implies $x \in B(p,r)$. □

Problem 2. In $\mathbb{R}$ show

$$\bigcap_{n=1}^{\infty} (0, 1/n) = \varnothing.$$

Problem 3. Let $r_1, r_2 > 0$ and $p \in E$ where E is a metric space. Show

$$B(p,r_1) \cup B(p,r_2) = B(p, \max(r_1, r_2))$$

and

$$B(p,r_1) \cap B(p,r_2) = B(p, \min(r_1, r_2))$$ □