Review for test 2

The one new thing that will be on the test are the axioms of projective. The basic notations of projective geometry are ***points***, ***lines*** and ***incidence*** between points and lines.

Axiom 1 (First Axiom of Projective Geometry)**.** *For any pair of distinct points there is unique line incident with these points.* □

Axiom 2 (Second Axiom of Projective Geometry)**.** *For any pair of distinct lines there is unique point incident with these lines.* □

Axiom 3 (Thrid Axiom of Projective Geometry)**.** *There are four points no three of which are incident with the same line.* □

The rest of the rest will mostly be based on Section 3–6 of Homework 3 (in the form currently on the class web page.) But, this being a math class, things are always cumulative and stuff from earlier sections may come up.

The main result from the last test (and Homework 2) was that if we let $\mathbb{A}^2$ be the set of ordered pairs (x, y) with $x, y \in \mathbb{F}$ where $\mathbb{F}$ is a our base field, and defined lines as the subsets of $\mathbb{A}^2$ given by

$$L(a, b, c) = \{(x, y) : ax + by + c = 0\}$$

where at least one of a or b is not zero, then this satisfied the three axioms of affine geometry (which you should still know). We have since then looked at some of the geometry. Here are some of the main ideas.

(1) **Affine combinations of points.** Given points $P_1, \ldots, P_n \in \mathbb{A}^2$ than an ***affine combination*** of this points is a sum

$$\alpha_1 P_1 + \cdots + \alpha_n P_n$$

where the scalars $\alpha_1, \ldots, \alpha_n$ satisfy

$$\alpha_1 + \cdots + \alpha_n = 1.$$

A special case of this is the ***midpoint*** of P_1 and P_2 which is

$$M = \frac{1}{2} P_1 + \frac{1}{2} P_2.$$

More generally the ***center of mass*** of $P_1, \ldots, P_n \in \mathbb{A}^2$ is

$$M = \frac{1}{n} P_1 + \cdots + \frac{1}{n} P_n.$$

Sample Problem. Let A, B, and C be points and

$$\begin{aligned} M_1 &= \text{Midpoint of } A \text{ and } B \\ M_2 &= \text{Midpoint of } A \text{ and } C \\ M_3 &= \text{Midpoint of } B \text{ and } C. \end{aligned}$$

Show

$$\text{Center of mass of } M_1,\ M_2,\ M_3 = \text{Center of mass of } A,\ B,\ C.$$

□

Sample Problem. Let A, B, and C be three points not all on the same line. Make a picture showing A, B, and C and the points

$$\frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C, \qquad 4A - 2B - C.$$

Hint: One way to think of this is that all non-collinear triples of points are affinely equivalent so we can choose A, B and C to be any points we wish. What would be a convenient triple of points to work with? □

Sample Problem. Problem 13 on Homework 3.

(2) **Affine parameterization of line.** The basic fact here is that if A and B are distinct points, then the set of point on $\overleftrightarrow{AB}$ is the same as the set of affine combinations of A and B. That is C on $\overleftrightarrow{AB}$ if and only

$$C = \alpha A + \beta B$$

where

$$\alpha + \beta = 1.$$

Half the proof of this is easy. That is if A and B are on the line $L(a, b, c)$, then any affine combination of A and B is on $L(a, b, c)$ is on $L(a, b, c)$. You should be able to prove this.

Sample Problem. Find the point where the line through $A = (1, 2)$ and $B = (3, 4)$ intersects the line $L(1, 2, 3)$. *Solution:* A point, P, on $\overleftrightarrow{AB}$ is of the form

$$P = (1 - t)A + tB = (1 - t)(1, 2) + t(3, 4) = (2t + 1, 2t + 2).$$

For this to be on $L(1, 2, 3)$ we must have

$$1(2t + 1) + 2(2t + 2) + 3 = 0.$$

Solving for t gives

$$t = -\frac{4}{3}$$

and therefore the point of intersection is

$$P = (2t + 1, 2t + 2)\Big|_{t=-4/3} = (-5/3, -2/3).$$

□

(3) **Affine maps.** A map $f \colon \mathbb{A}^2 \to \mathbb{A}^2$ is affine iff for all $P, Q \in \mathbb{A}^2$ and scalars α and β with $\alpha + \beta = 1$ we have

$$f(\alpha P + \beta Q) = \alpha f(P) + \beta f(Q).$$

We have shown that this implies that f does the right thing by all affine combinations. That is if $\alpha_1, \ldots, \alpha_n$ are scalars with

$$\alpha_1 + \cdots + \alpha_n = 1$$

and $P_1, \ldots, P_n$ are any points, then

$$f(\alpha_1 P_1 + \cdots + \alpha_n P_n) = \alpha_1 f(P_1) + \cdots + \alpha_n f(P_n).$$

Sample Problem. Problems 14, 17, 28 on Homework 3. $\square$

Sample Problem. If $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ is an affine bijection and ℓ and m are lines, then $m \parallel \ell$ if and only if $f[m] \parallel f[\ell]$. $\square$

Our main existence result about affine maps is the following.

Theorem 1. *If A is a 2×2 matrix and $\vec{b}$ is a vector, then*

$$F_{A,\vec{b}}(P) = AP + \vec{b}$$

is an affine map. $\square$

You should be able to prove this using the basic properties of matrices.

Sample Problem. Let $P = (p_1, p_2)$, $Q = (q_1, q_2)$, $R = (r_1, r_2)$ be points in $\mathbb{A}^2$. Show there is an affine map $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ with

$$f(0,0) = P, \qquad f(1,0) = Q, \qquad f(0,1) = R.$$

Solution. We will find A and $\vec{b}$ so that $f = F_{A,\vec{b}}$ does the trick. We first find the matrix A such that

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = Q - P = \begin{bmatrix} q_1 - p_1 \\ q_2 - p_2 \end{bmatrix} \qquad \text{and} \qquad A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = R - P = \begin{bmatrix} r_1 - p_1 \\ r_2 - p_2 \end{bmatrix}.$$

That is

$$A = \begin{bmatrix} q_1 - p_1 & r_1 - p_1 \\ q_2 - p_2 & r_2 - p_2 \end{bmatrix}.$$

Set

$$\vec{b} = P.$$

We check that this works.

$$F_{A,P}(0,0) = A\vec{0} + P = P,$$

$$F_{A,P} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} + P = Q - P + P = Q,$$

and

$$F_{A,P} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A \begin{bmatrix} 0 \\ 1 \end{bmatrix} + P = R - P + P = R.$$

and we are done. $\square$

(4) **Affinely independent sets.** A set of three points $P, Q, R \in \mathbb{A}^2$ is ***affinely independent*** iff they do not all lie on a line. That is they are not collinear.

Here are some basic results about affine independent sets you should know.

Theorem 2. *If A, B, and C are affinely independent points, then point P can be uniquely expressed as an affine combination of these points.*

Sample Problem. Use the last theorem to show that if two affine maps, f and g, agree on on a set of three affinely independent points then $f = g$. $\square$

Theorem 3. *If A, B, and C are affinely independent, then for any three points A', B', and C' there is a unique affine map $f \colon \mathbb{A}^2 \to \mathbb{A}^2$ with $f(A) = A'$, $F(B) = B'$ and $f(C) = C'$.* $\square$

Theorem 4. *Any two triangles are affinely equivalent.*

Sample Problem. Use Theorem 3 to prove Theorem 4. $\square$

Sample Problem. Problems 25, 29, 40 on Homework 3.

(5) **Desargues's Theorem.** Know the statement of this result.
(6) Surprise mystery questions.