Mathematics 555 Test 1 Name: **Answer Key.**

1. (a) Let $f\colon E \to E'$ be a map between metric spaces. Define f is ***uniformly continuous***.

Solution: For all $\varepsilon > 0$ there is a $\delta > 0$ such that if $p, q \in E$ with $d(p,q) < \delta$, then $d(f(p), f(q)) < \varepsilon$. $\square$

(b) Give an example of a function $f\colon (0,1) \to \mathbf{R}$ that is continuous, but not uniformly continuous. (You do not have to prove that your example works.)

Solution: One of the examples at least one of you gave on a homework was

$$f(x) = \frac{1}{x}.$$

If you want a bounded example then

$$f(x) = \sin(1/x)$$

does the trick. $\square$

2. (a) Let $f\colon (a,b) \to \mathbf{R}$ and let $x_0 \in (a,b)$. Define what if means for f to be ***differentiable*** at x_0.

Solution: The function f is differentiable at x_0 if and only if the following limit exists:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

The value of this limit is $f'(x_0)$, the ***derivative*** of f at x_0. $\square$

(b) Let $f\colon (0,\infty) \to \mathbf{R}$ be given by

$$f(x) = \frac{1}{x^2}.$$

Prove directly from the definition that f is differentiable at 2 and find $f'(2)$.

Solution: We compute the limit defining $f'(2)$ by simplifying the difference quotient in the definition of the derivative:

$$\begin{aligned}
f'(2) &= \lim_{x \to x_0} \frac{f(x) - f(2)}{x - 2} \\
&= \lim_{x \to 2} \frac{1}{x - 2} \left(\frac{1}{x^2} - \frac{1}{2^2} \right) \\
&= \lim_{x \to 2} \frac{1}{x - 2} \left(\frac{2^2 - x^2}{2^2 x^2} \right) \\
&= \lim_{x \to 2} \frac{1}{x - 2} \left(\frac{(2 - x)(2 + x)}{2^2 x^2} \right) \\
&= \lim_{x \to 2} \frac{-(x + 2)}{4x^2} \\
&= \lim_{x \to 2} \frac{-(2 + 2)}{4(2)^2} \\
&= \frac{-1}{4}.
\end{aligned}$$

□

3. Let $f, f_1, f_2, f_3, \ldots : [0, 1] \to \mathbf{R}$ be functions.

(a) Define what it means for $\lim_{n\to\infty} f_n = f$ ***pointwise***.

Solution: Either of the following is acceptable:

(i) For all $x \in [0, 1]$ we have $\lim_{n\to\infty} f_n(x) = f(x)$.

(ii) For all $x \in [0, 1]$ and $\varepsilon > 0$ there is a N such that $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon$.

(b) Define what is means for $\lim_{n\to\infty} f_n = f$ ***uniformly***.

Solution: For all $\varepsilon > 0$ there is a N such that $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon$ for all $x \in [0, 1]$.

(c) If each of $f_1, f_2, f_2, \ldots$ is continuous and $\lim_{n\to\infty} f_n = f$ uniformly, what can be said about f?

Solution: One of our theorems is that the uniform limit of continuous functions is continuous. Thus f is continuous.

(d) Give an example of functions $f, f_1, f_2, f_3, \ldots : [0, 1] \to \mathbf{R}$ so that $\lim_{n\to\infty} f_n = f$ pointwise, each f_k is continuous, but f is not continuous at the point $1/2$.

Solution: There are many solutions. One is to let

$$g(x) = (1 - |x - 1/2|)$$

on $[0,1]$. Then $g(1/2) = 1$ and $0 \le g(x) < 1$ for all $x \in [0,1]$ other than $x = 1/2$. Set

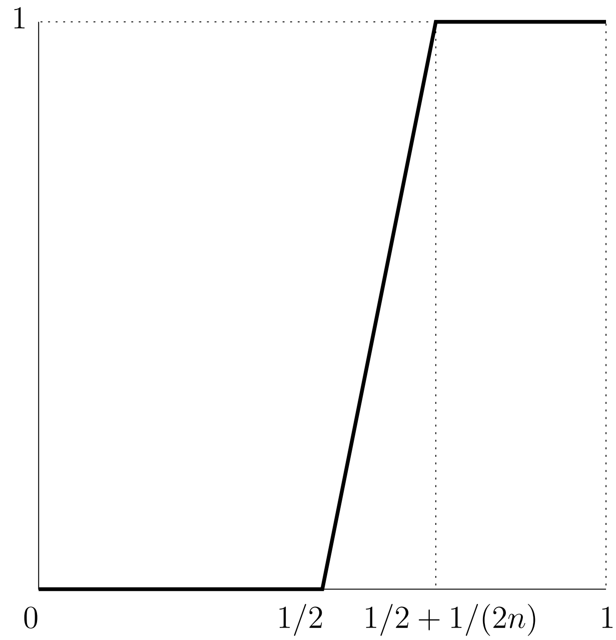
$$f_n(x) = g(x)^n = (1 - |x - 1/2|)^n$$

Then for $x \in [0,1]$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} g(x)^n = f(x) = \begin{cases} 1, & x = 1/2; \\ 0, & x \ne 1/2. \end{cases}$$

The functions $f_n(x)$ are continuous and $f(x)$ is not.

It is probably easier to give an example by drawing the graphs. Here is one such example. Let f_n have the graph as shown



and let

$$f(x) = \begin{cases} 0, & 0 \le x \le 1/2; \\ 1, & 1/2 < x \le 1. \end{cases}$$

Then each f_n is continuous, $\lim_{n\to\infty} f_n = f$ pointwise, but f is not continuous.

(e) Explain why the in part (d) the convergence can not be uniform.

Solution: If the convergence were continuous, the limit would be continuous. $\square$

4. (a) State the Mean Value Theorem.

Solution: Let $f\colon [a,b] \to \mathbf{R}$ be continuous on $[a,b]$ and differentiable on the open interval (a,b). Then there is a points $\xi \in (a,b)$ with

$$f(b) - f(a) = f'(\xi)(b-a).$$

(b) Let $f\colon \mathbf{R} \to \mathbf{R}$ such that f is differentiable at all points and with

$$f'(x) = \frac{1+f(x)^2}{2+f(x)^2}$$

for all x. Explain why f is increasing.

Solution: As $(1+f(x)^2)/(2+f(x)) > 0$ we see that f' is positive. Therefore f is increasing.

(c) With f is in part (b) show

$$|f(b) - f(a)| \leq |b-a|$$

for all $a, b \in \mathbf{R}$.

Solution: We have

$$0 \leq f'(x) = \frac{1+f(x)^2}{2+f(x)^2} < \frac{2+f(x)^2}{2+f(x)^2} = 1.$$

Therefore we can use the Mean Value Theorem to conclude there is a ξ between a and b with

$$|f(b) - f(a)| = |f'(\xi)||b-a| \leq 1|b-a| = |b-a|.$$

$\square$

5. (a) State Taylor's Theorem with Lagrange's form of the remainder.

Solution: Let $f\colon (a,b) \to \mathbf{R}$ be $n+1$ times differentiable and let $x_0 \in (a,b)$. Let

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

be the degree n Taylor's polynomial for f at x_0. Then for $x \in (a,b)$ there is a ξ between x_0 and x such that

$$f(x) = T_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}.$$

$\square$

(b) Let $h\colon \mathbf{R} \to \mathbf{R}$ be a twice differentiable function with

$$f(0) = 1, \quad f'(0) = -1, \quad \text{and} \quad f''(x) \geq 2 \text{ for all } x \in \mathbf{R}.$$

Prove for all $x \in \mathbf{R}$

$$f(x) \geq 1 - x + x^2.$$

Solution: By Taylor's Theorem with $n = 1$ and $x_0 = 0$ and $x \in \mathbf{R}$ we have that there is a ξ between 0 and x such that

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(\xi)}{2}x^2 \\ &= 1 - x + \frac{f''(\xi)}{2}x^2 && \text{(as } f(0) = 1 \text{ and } f'(0) = -1) \\ &\geq 1 - x + \frac{2}{2}x^2 && \text{(as } f''(\xi) \geq 2) \\ &= 1 - x + x^2 \end{aligned}$$

as required. $\square$