

Math 555 Homework.

Here are some problems for over Spring break.

Problem 1. First read Sections 1, 2, and 3 of Homework 4 (that is up through the Integral Test) and do Problems 1–5. \square

Problem 2. Compute the following limits:

(a) $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{\infty} \left(\frac{1}{2k}\right)^3.$

(b) $\lim_{n \rightarrow \infty} \sum_{k=n}^{4n} \frac{1}{k+n}.$

(c) $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0.$ *Hint:* For $x > 0$ we have $\sqrt{x} < x$ and thus

$$\ln(n) = \int_1^n \frac{dx}{x} < \int_1^n \frac{dx}{\sqrt{x}} = 2\sqrt{n} - 2.$$

(d) For $\alpha > 0$, $\lim_{n \rightarrow \infty} \frac{\ln n}{n^\alpha} = 0.$ That is $\ln(n)$ grows more slowly than any power of $n.$ *Hint:*

$$\frac{\ln n}{n^\alpha} = \frac{\ln(n^\alpha)}{\alpha n^\alpha}$$

and use the limit of part (c). \square

Problem 3. (a) Let k be a positive integer. Show for all $x > 0$

$$e^x > \frac{x^k}{k!}.$$

Hint: Taylor's Theorem.

(b) Let $r, \alpha > 0.$ Show

$$\lim_{n \rightarrow \infty} \frac{n^\alpha}{e^{rn}} = 0.$$

That is e^{rn} grows faster than any power of $n.$ *Hint:* From Part (a) we have that for any positive integer $k,$

$$e^{rn} \geq \frac{r^k n^k}{k!}$$

and so

$$0 < \frac{n^\alpha}{e^{rn}} < \frac{k! n^\alpha}{r^k n^k}.$$

We can use any k we want, so choose $k > \alpha.$ \square

Problem 4. In this problem we show that the integration by parts trick we used to prove Taylor's Theorem can be used to prove other interesting

results. Let k be an integer and let

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x - (k + 1/2) \\ P_2(x) &= \frac{(x - k)(x - k - 1)}{2}. \end{aligned}$$

Let $f(x)$ be a function on $[k, k + 1]$ that is twice continuously differentiable.

(a)

$$\begin{aligned} P_1'(x) &= 1 \\ P_2'(x) &= P_1(x). \end{aligned}$$

(b) Integrate

$$\int_k^{k+1} f(x) dx = \int_k^{k+1} P_1'(x) f(x) dx$$

by parts twice to get

$$\int_k^{k+1} f(x) dx = P_1(x)f(x) \Big|_k^{k+1} - P_2(x)f'(x) \Big|_k^{k+1} + \int_k^{k+1} P_2(x)f''(x) dx.$$

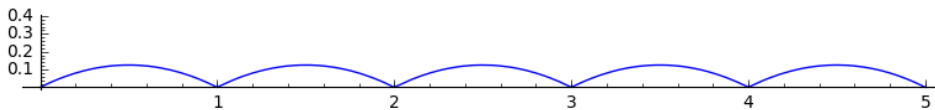
(c) Show this simplifies down to

$$\int_k^{k+1} f(x) dx = \frac{f(k) + f(k+1)}{2} + \int_k^{k+1} P_2(x)f''(x) dx$$

(d) Now define a function $B: \mathbb{R} \rightarrow \mathbb{R}$ by

$$B(x) = \frac{(x - k)(k + 1 - x)}{2} = -P_2(x) \quad \text{when } k \leq x \leq k + 1.$$

This function is periodic with period 1, that is $B_2(x+1) = B(x)$. The graph looks like



Show that with this notation we have

$$\frac{f(k) + f(k+1)}{2} = \int_k^{k+1} f(x) dx + \int_k^{k+1} B(x)f''(x) dx.$$

Also show

$$0 \leq B(x) \leq \frac{1}{8}.$$

(The advantage of working with $B(x)$ is that it is positive, and so it is easier to keep track of signs.)

(e) Now sum the equality for $(f(k) + f(k+1))/2$ from $k = 1$ to $n - 1$ and rearrange a bit to get

$$(1) \quad \sum_{k=1}^n f(k) = \int_1^n f(x) dx + \frac{f(1) + f(n)}{2} + \int_1^n B(x) f''(x) dx.$$

This gives a precise relation between sums and integrals of the same function.

Problem 5. To give a concrete application of the formula (1) we let $f(x) = \ln(x)$ and derive a very useful approximation to $n!$ (*Stirling's Formula*).

(a) With this choice of $f(x)$ show that (1) becomes

$$\begin{aligned} \ln(n!) &= \sum_{k=1}^n \ln(k) \\ &= \int_1^n \ln(x) dx + \frac{\ln(1) + \ln(n)}{2} + \int_1^n B(x) \ln''(x) dx \\ &= n \ln(n) - n + \frac{\ln(n)}{2} - \int_1^n \frac{B(x)}{x^2} dx \end{aligned}$$

(b) Note that

$$0 < \int_1^n \frac{B(x)}{x^2} dx < \int_1^n \frac{1}{8x^2} dx = \frac{1}{8} \left(1 - \frac{1}{n}\right) < \frac{1}{8}$$

and use this to show that

$$C := \lim_{n \rightarrow \infty} \int_1^n \frac{B(x)}{x^2} dx$$

exists and

$$0 < C \leq \frac{1}{8}.$$

(c) Rewrite the formula for $\ln(n!)$ as

$$\ln(n!) = n \ln(n) - n + \frac{\ln(n)}{2} + C - R_n$$

where

$$R_n = C - \int_0^n \frac{B(x)}{x^2} dx.$$

satisfies

$$0 < R_n \leq \frac{1}{8n}.$$

(d) Use this to conclude

$$n! = e^C n^{n+1/2} e^{-n} e^{-R_n}.$$

Setting $K = e^C$ we then have

$$K n^{n+1/2} e^{-n} e^{-1/(8n)} < n! < K n^{n+1/2} e^{-n}.$$

Or in slightly different form

$$e^{-1/(8n)} < \frac{n!}{Kn^{n+1/2}e^{-n}} < 1.$$

Thus $n!$ has the same growth rate as $Kn^{n+1/2}e^{-n} = K\sqrt{n}\left(\frac{n}{e}\right)^n$. Of course we would like to know the constant K . It turns out

$$K = \sqrt{2\pi},$$

a fact that hopefully we will be able to show by the end of the term. Using this we can rewrite our inequalities as

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{-1/8n} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n. \quad \square$$