

More about Series.

Consider two power series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$g(x) = \sum_{k=0}^{\infty} b_k x^k$$

If we assume that we can multiply these the same way we would polynomials we get

$$\begin{aligned} f(x)g(x) &= (a_0 + a_1x + a_2x^2 + a_3x^3 + \dots)(b_0 + b_1x + b_2x^2 + b_3x^3 + \dots) \\ &= a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \dots \\ &= \sum_{k=0}^{\infty} \left(\sum_{j=0}^k a_j b_{k-j} \right) x^k \end{aligned}$$

Problem 1. Here is another way to see this. Let

$$h(x) = f(x)g(x).$$

Then the first couple of derivatives of h are

$$\begin{aligned} h'(x) &= f'(x)g(x) + f(x)g'(x) \\ h''(x) &= f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x) \\ h'''(x) &= f'''(x)g(x) + 3f''(x)g'(x) + 3f'(x)g''(x) + g'''(x) \end{aligned}$$

which reminds us of the Binomial Theorem.

(a) Prove that k -th derivative of $h(x)$ is

$$h^{(k)}(x) = \sum_{j=0}^k \binom{k}{j} f^{(j)}(x) g^{(k-j)}(x).$$

(b) Let

$$a_k = \frac{f^{(k)}(0)}{k!} \quad \text{and} \quad b_k = \frac{g^{(k)}(0)}{k!}$$

and use the formula for $h^{(k)}(0)$ to show

$$\frac{h^{(k)}(0)}{k!} = \sum_{j=0}^k a_j b_{k-j}. \quad \square$$

If we assume that both series for $f(x)$ and $g(x)$ both converge for $x = 1$ we can let $x = 1$ the result is

$$\left(\sum_{k=0}^{\infty} a_k \right) \left(\sum_{n=0}^{\infty} b_n \right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^k a_j b_{k-j} \right).$$

This motivates:

Definition 1. Let

$$\sum_{k=0}^{\infty} a_k \quad \sum_{k=0}^{\infty} b_k$$

be two series then the **Cauchy product** of these series is the series

$$\sum_{k=0}^{\infty} c_k$$

where

$$c_k = \sum_{j=0}^k a_j b_{k-j} = \sum_{i+j=k} a_i b_j. \quad \square$$

Theorem 2. Let

$$A = \sum_{k=0}^{\infty} a_k \quad B = \sum_{k=0}^{\infty} b_k$$

we convergent series with at least one of the two absolutely convergent. Let

$$c_k = \sum_{j=0}^k a_j b_{k-j}.$$

Then the series $\sum_{k=0}^{\infty} c_k$ converges and

$$\sum_{k=0}^{\infty} c_k = AB.$$

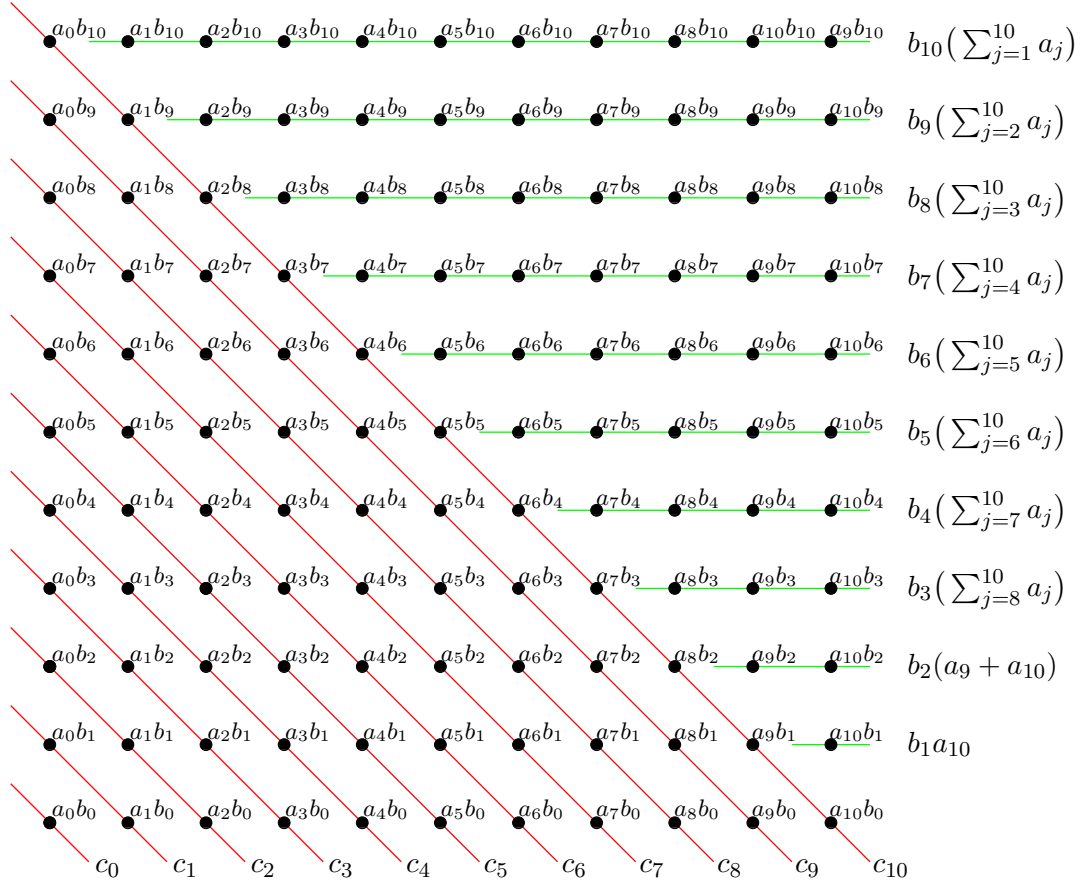
Problem 2. Prove this along the following lines. Let

$$A_n = \sum_{k=0}^n a_k \quad B_n = \sum_{k=0}^n b_k \quad C_n = \sum_{k=0}^n c_k$$

be the partial sums. Note that the product

$$A_n B_n = (a_0 + a_1 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n)$$

is the sum of the $(n+1)^2$ products $a_j b_k$ with $0 \leq j, k \leq n$. For $n = 10$ these terms are shown in the following figure.



(a) Using the figure above as a guide show for all $n = 1, 2, \dots$ that

$$A_n B_n - C_n = \sum_{k=1}^n b_k \left(\sum_{j=n-k+1}^n a_j \right).$$

(b) Let $0 < m < n$ (the integer m will be chosen later). Explain why

$$\begin{aligned} |A_n B_n - C_n| &= \left| \sum_{k=1}^n b_k \left(\sum_{j=n-k+1}^n a_j \right) \right| \\ &\leq \sum_{k=1}^n |b_k| \left| \sum_{j=n-k+1}^n a_j \right| \\ &= \sum_{k=1}^m |b_k| \left| \sum_{j=n-k+1}^n a_j \right| + \sum_{k=m+1}^n |b_k| \left| \sum_{j=n-k+1}^n a_j \right| \end{aligned}$$

- (c) Without loss of generality we may assume that $\sum_{k=0}^{\infty} b_k$ is absolutely convergent. Explain why there is constant $\beta \geq 0$ such that for all m

$$\sum_{k=1}^m |b_k| \leq \beta.$$

- (d) The series $\sum_{j=1}^{\infty} a_j$ is convergent. That is $\lim_{n \rightarrow \infty} A_n$ exists. Show this implies there is a constant C such that $|A_n| \leq C$ for all n and then use

$$\sum_{j=n-k+1}^n a_j = A_n - A_{n-k}$$

to show there is a constant $\alpha \geq 0$ such that

$$\left| \sum_{j=n-k+1}^n a_j \right| \leq \alpha$$

for all n and k with $0 \leq k \leq n$.

- (e) Combine parts (b), (c), and (d) to show

$$\begin{aligned} |A_n B_n - C_n| &\leq \beta \left| \sum_{j=n-k+1}^n a_j \right| + \alpha \sum_{k=m+1}^n |b_k| \\ &= \beta |A_n - A_{n-k}| + \alpha \sum_{k=m+1}^n |b_k| \end{aligned}$$

when $0 \leq k \leq m \leq n$.

- (f) Let $\varepsilon > 0$. Explain where are are $N_1, N_2 > 0$ such that

$$m \geq N_1 \quad \text{implies} \quad \sum_{k=m+1}^n |b_k| < \frac{\varepsilon}{2\alpha},$$

and

$$n \geq N_2 \quad \text{and} \quad n - k \geq N_2 \quad \text{implies} \quad |A_n - A_{n-k}| < \frac{\varepsilon}{2\beta}.$$

- (g) Let $n \geq N_1 + N_2 + 2$, set $m = N_1$, and show that for any k with $0 \leq k \leq m$ that the inequalities

$$m \geq N_1, \quad n \geq N_2, \quad n - k \geq N_2$$

all hold and that this in turn yields

$$n \geq N_1 + N_2 + 2 \quad \text{implies} \quad |A_n B_n - C_n| < \varepsilon.$$

- (h) Conclude from part (f) that

$$\lim_{n \rightarrow \infty} (A_n B_n - C_n) = 0.$$

- (i) Complete the proof by showing

$$\lim_{n \rightarrow \infty} C_n = AB.$$

□

Problem 3. Here is an example to show that it is important that in Theorem 2 at least one of the two series is absolutely convergent. Let $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} (-1)^k / \sqrt{k+1}$. The Cauchy product is $\sum_{k=0}^{\infty} c_k$ where

$$c_k = (-1)^k \sum_{j=0}^k \frac{1}{\sqrt{(j+1)(k-j+1)}}.$$

Show

$$|c_k| = \sum_{j=0}^k \frac{1}{\sqrt{(j+1)(k-j+1)}} \geq 1$$

and therefore the series $\sum_{k=0}^{\infty} c_n$ diverges. \square

Theorem 3. *Let*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \quad g(x) = \sum_{k=0}^{\infty} b_k x^k$$

be power series with radius convergence at least R . Let

$$c_n = \sum_{j=0}^n a_j b_{n-j}$$

then the power series

$$h(x) = \sum_{n=0}^{\infty} c_n x^n$$

also has radius of convergence at least R and

$$h(x) = f(x)g(x)$$

for $|x| < R$.

Proof. This follows easily from Theorem 2. \square

We now give a short indication of how to divide power series. Assume that we wish to find the power series expansion of

$$f(x) = \frac{h(x)}{g(x)}$$

where

$$h(x) = \sum_{k=0}^{\infty} c_k x^k \quad g(x) = \sum_{k=0}^{\infty} b_k x^k.$$

and we wish to find the series for $f(x)$. Assume that $f(x)$ has an expansion

$$f(x) = \sum_{k=0}^{\infty} a_k x^k.$$

Then $h(x) = f(x)g(x)$ and so we have

$$c_k = \sum_{j=0}^k a_j b_{k-j}.$$

Assume that $g(0) \neq 0$, that is $b_0 \neq 0$. Then the last equation can be rewritten as

$$a_k = \frac{1}{b_0} \left(c_k - \sum_{j=0}^{k-1} a_j b_{k-j} \right).$$

For small values of k these formulas are

$$\begin{aligned} a_0 &= \frac{c_0}{b_0} \\ a_1 &= \frac{1}{b_0} (c_1 - a_0 b_1) \\ a_2 &= \frac{1}{b_0} (c_2 - a_0 b_2 - a_1 b_1) \\ a_3 &= \frac{1}{b_0} (c_3 - a_0 b_3 - a_1 b_2 - a_2 b_1) \\ a_4 &= \frac{1}{b_0} (c_4 - a_0 b_4 - a_1 b_3 - a_2 b_2 - a_3 b_1) \end{aligned}$$

This allows us to find the coefficients a_0, a_1, a_2, \dots of $f(x)$ recursively. Unfortunately this method does not tell us anything about the radius of convergence of $f(x)$ in terms of the radii of convergence of $g(x)$ and $h(x)$. But if we already know that all three have positive radius of convergence, it does give us a method for finding the coefficients of $f(x)$ from the coefficients of $g(x)$ and $h(x)$.

Problem 4. Find the first three nonzero terms in the power series of

$$f(x) = \frac{e^{2x}}{\cos(x)}. \quad \square$$

Problem 5. Find the first couple terms of the power series of the following and thus convince yourself that using series tells you more than using L'Hôpital's rule.

- (a) $\frac{\sin(2x)}{4x}$
- (b) $\frac{1 - \cos(5x)}{x^2}$
- (c) $\frac{e^x - 1 - x}{1 - \cos(2x)}.$ □

Problem 6. Find the power series of the following functions around the indicated points x_0 .

- (a) $f(x) = \sin(x)$ around $x_0 = \pi/4$.
- (b) $f(x) = e^{2x}$ around the point $x_0 = 1$.
- (c) $f(x) = \sqrt{4-x}$ around the point $x = 4$.