Even more about Series.

Here we look at power series centered at points other than the origin. To start we generalize a result we used as lemma when working with power series.

Lemma 1. *Let*

$$p(x) = b_t x^t + b_{t-1} t^{t-1} + \cdots b_0$$

be a polynomial, r a real number with $|r| < 1$, and $\ell \geq 0$ an integer. Then the series

$$\sum_{k=\ell}^{\infty} p(k) r^{k-\ell}$$

converges.

Problem 1. Prove this. *Hint:* Use the ratio test. □

We are now going to consider power series of the form

$$f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k.$$

We are already experts on the special case where $x_0 = 0$, and this case can be reduced to that case by the change of variable $y = x - x_0$, by let us redo the theory as a review.

Theorem 2. *Let*

$$f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k.$$

and assume that this series converges at the point $x = x_1$. Then for any x closer to x_0 than x_1, that is with $|x - x_0| < |x - x_0| < |x_1 - x_0|$, then $f(x)$ converges absolutely at x.

Proof. As the series

$$\sum_{k=0}^{\infty} c_k (x_1 - x_0)^k$$

converges there is a constant B such that

$$|c_k (x_1 - x_0)^k| \leq B.$$

We use what has become a standard trick:

$$|c_k (x - x_0)^k| = \left| c_k (x_1 - x_0)^k \frac{(x - x_0)^k}{(x_1 - x_0)^k} \right| \leq B r^k$$

where

$$r = \left| \frac{x - x_0}{x_1 - x_0} \right| < 1$$

thus the series for $f(x)$ converges absolutely by comparison to the geometric series $\sum_{k=0}^{\infty} B r^k$. □

Definition 3. Given the series

$$f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$$

the number

$$R = \sup \big\{ |x_1 - x_0| : f(x_1) \text{ converges.} \big\}$$

is the ***radius of convergence*** of $f(x)$. □

As we have seen in the case where $x_0 = 0$ there are example where $R = 0$ and $R = \infty$.

Proposition 4. *Let $f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$ have radius of convergence R. Then $f(x)$ converges absolutely on the open interval $(x_0 - R, x_0 + R)$ and diverges outside of the closed interval $[x_0 - R, x_0 + R]$. The series may or may not converge at the endpoints $x_0 - R$ and $x_0 + R$ depending on the series.*

Proof. This follows from Theorem 2 and the definition of the radius of convergence. □

Proposition 5. *Let $f_k \colon [x_0 - r, x_0 + r] \to \mathbf{R}$ be a continuous function for $k = 0, 1, 2, \ldots$. Assume the series*

$$f(x) = \sum_{k=0}^{\infty} f_k(x)$$

converges uniformly on $[x_0 - r, x_0 + r]$. Then the series

$$F(x) = \sum_{k=0}^{\infty} \int_{x_0}^{x} f_k(t)\, dt$$

also converges uniformly on $[x_0 - r, x_0 + r]$ and

$$F(x) = \int_{x_0}^{x} f(x)\, dt.$$

Informally this tells us that for a uniformly convergent series of functions we can integrate term wise:

$$\int_{x_0}^{x} \left(\sum_{k=0}^{\infty} f_k(t) \right) dt = \sum_{k=0}^{\infty} \int_{x_0}^{x} f_k(t)\, dt.$$

Proof. Let

$$s_n(x) = \sum_{k=0}^{n} f_k(x)$$

be the n-th partial sum of the series for $f(x)$. Then s_n is a finite sum of continuous functions and s_n converges to f uniformly and therefore f is continuous. Let

$$F_n(x) = \int_{x_0}^{x} s_n(x).$$

Then, as the sum is finite,

$$F_n(x) = \int_{x_0}^{x} \left(\sum_{k=0}^{n} f_k(t) \right) dt = \sum_{k=0}^{n} \int_{x_0}^{x} f_k(t)\, dt.$$

Therefore F_n is the n-th partial sum for the series for F.

Let $\varepsilon > 0$. As $s_n \to f$ uniformly there is N such that

$$|s_n(x) - f(x)| < \frac{\varepsilon}{r}$$

for all $x \in [x_0 - r, x_0 + r]$. Then for $n \geq N$ and $x \in [x_0 - r, x_0 + r]$

$$\begin{aligned}\left| \int_{x_0}^{x} f(t)\, dt - F_n(x) \right| &= \left| \int_{x_0}^{x} f(t)\, dt - \int_{x_0}^{x} s_n(t)\, dt \right| \\ &\leq \left| \int_{x_0}^{x} |f(t) - s_n(t)|\, dt \right| \\ &< \left| \int_{x_0}^{x} \frac{\varepsilon}{r} dt \right| \\ &= \frac{\varepsilon}{r} |x - x_0| \\ &\leq \varepsilon.\end{aligned}$$

This shows that

$$\lim_{n \to \infty} F_n(x) = \int_{x_0}^{x} f(t)\, dt.$$

But $F_n(x)$ is the partial sum for the series $\sum_{k=0}^{\infty} f_k(t)\, dt$, which completes the proof. □

Proposition 6. *Let* $f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$ *have radius of convergence* R*. Then for every* r *with* $0 < r < R$ *the series for* $f(x)$ *and the series for the formal derivative*

$$f^*(x) = \sum_{k=1}^{\infty} k c_k (x - x_0)^{k-1}$$

converges uniformly and absolutely on the interval $[x_0 - r, x_0 + r]$*.*

Proof. Choose r_1 with $r < r_1 < R$. Then the series for $f(x)$ converges at the point $x = x_0 + r_1$ (as $|x - x_0| = r_1 < R$ and thus

$$\sum_{k=0}^{\infty} c_k r_1^k$$

converges. It follows that there is a constant B such that

$$|c_k r_1^k| \leq B.$$

Let

$$\rho = \frac{r}{r_1}.$$

Then $0 < \rho < 1$. And by our multiplying and dividing trick we have for $x \in [x_0 - r, x_0 + r]$ that

$$\begin{aligned} \left|c_k(x-x_0)^k\right| &= \left|c_k r_1^k \frac{(x-x_0)}{r_1}\right| \le B\rho^k \\ \left|kc_k(x-x_0)^{k-1}\right| &= \left|kc_k r_1^{k-1} \frac{(x-x_0)^{k-1}}{r_r^{k-1}}\right| \le kB\rho^{k-1}. \end{aligned}$$

Let $M_k = B\rho^k$ in the Weierstrass M-test shows that the series $f(x)$ converges absolutely and uniformly on $[x_0 - r, x_0 + r]$. Using the M-test with $M_k = kM\rho^{k-1}$ and Proposition 1. does the trick for the series for $f^*(x)$. □

Theorem 7. *Assume that the series $f(x) = \sum_{k=0}^{\infty} c_k(x - x_0)^k$ has radius of convergence R. Then f is differentiable on the interval $(x_0 - R, x_0 + R)$ and the derivative of f is given by*

$$f'(x) = \sum_{k=1}^{\infty} kc_k(x - x_0)^{k-1}.$$

Problem 2. Prove this. *Hint:* Let $f^*(x) = \sum_{k=1}^{\infty} kc_k(x - x_0)^{k-1}$, let $0 < r < R$, and apply Theorem 5 to f^*. □

Repeated application of Theorem 7 implies that f has derivative of all orders on $(x_0 - r, x_0 + r)$. We can now derive a formula for the coefficients, c_k, of f. By taking taking derivatives we get

$$\begin{aligned} f(x) &= c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + c_3(x-x_0)^3 + c_4(x-x_0)^4 + \cdots \\ f'(x) &= c_1 + 2c_2(x-x_0) + 3c_3(x-x_0)^2 + 4c_4(x-x_0)^3 + 5c_5(x-x_0)^4 + \cdots \\ f''(x) &= 2c_2 + 3\cdot 2c_3(x-x_0) + 4\cdot 3c_4(x-x_0)^2 + 5\cdot 4c_5(x-x_0)^3 + \cdots \\ f'''(x) &= 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-x_0) + 5\cdot 4\cdot 3c_5(x-x_0)^2 + \cdots \\ \vdots &= \vdots \\ f^{(k)}(x) &= k(k-1)(k-2)\cdots(2)(1)c_k + (k+1)(k)(k-1)\cdots(3)(2)c_{k+1}(x-x_0) + \cdots \end{aligned}$$

Evaluating at $x = x_0$ gives

$$\begin{aligned} f(x_0) &= c_0 \\ f'(x_0) &= c_1 \\ f''(x_0) &= 2c_2 \\ f'''(x_0) &= 6c_3 \\ \vdots &= \vdots \\ f^{(k)}(x_0) &= k!c_k \end{aligned}$$

This proves:

Theorem 8. *If* $f(x) = \sum_{k=0}^{\infty} c_k(x - x_0)^k$ *has a positive radius of convergence, then the coefficients are given by*

$$c_k = \frac{f^{(k)}(x_0)}{k!}.$$

□