Mathematics 551 Homework, January 15, 2020

We start by reviewing some vector algebra. Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ be vectors in $\mathbb{R}^2$ and c a scalar. Then the sum of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2)$$

and the product of $\mathbf{a}$ by the c is

$$c\mathbf{a} = (ca_1, cb_1).$$

It is common to use the notation

$$\mathbf{i} = (1, 0), \qquad \mathbf{j} = (0, 1).$$

With this notation we can write $\mathbf{a}$ as

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}.$$

The ***inner product*** of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2.$$

Then

$$\mathbf{a} \cdot \mathbf{a} = (a_1)^2 + (a_2)^2$$

which is the square of the length of $\mathbf{a}$. We use the notation

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

for the ***length*** of $\mathbf{a}$.

If c_1 and c_2 are scalars and $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors are scalars then the inner product has the following properties:

- $\mathbf{b} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.
- $(c_1\mathbf{u} + c_2\mathbf{v}) \cdot \mathbf{w} = c_1\mathbf{u} \cdot \mathbf{w} + c_2\mathbf{v} \cdot \mathbf{w}$.

Consequences of these that will come up are

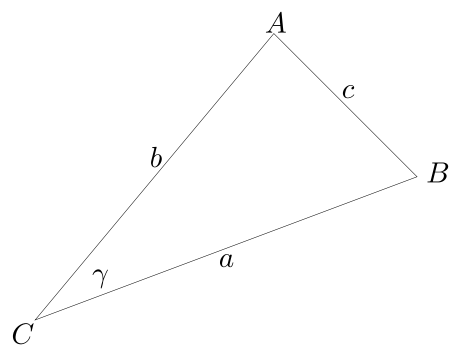
$$\|\mathbf{a} + \mathbf{b}\| = \|\mathbf{a}\|^2 + 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{a}\|^2$$
$$\|\mathbf{a} - \mathbf{b}\| = \|\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2.$$

Problem 1. Prove these formulas. □

A very important property of the inner product is given by

Theorem 1. *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are nonzero vectors and* θ *is the angle between them, then*

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos(\theta). \tag{1}$$



Problem 2. In the triangle shown a, b, and c are the side lengths of $\triangle ABC$ and γ is the angle between $\overrightarrow{CB}$ and and $\overrightarrow{CA}$. Use Theorem 1 to show

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma).$$

Hint: Let $\mathbf{u} = \overrightarrow{CB}$, $\mathbf{v} = \overrightarrow{CA}$ and $\mathbf{w} = \overrightarrow{AB}$. Then $\mathbf{w} = \mathbf{u} - \mathbf{v}$ and therefore $\|\mathbf{w}\|^2 = \|\mathbf{u} - \mathbf{v}\|^2$. □

A Corollary of Theorem 1 is that non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Problem 3. Show that for any vectors $\mathbf{a}$ and $\mathbf{b}$ that $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are perpendicular if and only if $\|\mathbf{a}\| = \|\mathbf{b}\|$. □

Problem 4. Define a map $J\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$J(x, y) = (-y, x).$$

Show for all non-zero vectors $\mathbf{v}$ that

(a) $\|J\mathbf{v}\| = \|\mathbf{v}\|$
(b) $\mathbf{v}$ and $J\mathbf{v}$ are always perpendicular. □

We now recall some calculus for functions $\mathbf{c}\colon [a, b] \to \mathbb{R}^2$ or more generally functions $\mathbf{c}\colon [a, b]\mathbb{R}^n$ where $\mathbb{R}^n$ is the space of n-tuples $(x_1, x_2, \ldots, x_n)$. We give the formulas for $n = 2$, and generalizing to higher dimensions is easy. First if

$$\mathbf{c}(t) = (x(t), y(t))$$

then the ***derivative*** of $\mathbf{c}(t)$ is

$$\mathbf{c}'(t) = (x'(t), y'(t)).$$

That is computing the derivative of the function $\mathbf{c}(t) = (x(t), y(t))$ is the same as computing the derivative of each component. The official definition is in terms of a limit;

$$\mathbf{c}'(t) = \lim_{h \to 0} \frac{1}{h} \left(\mathbf{c}(t + h) - \mathbf{c}(t) \right).$$

With this definition some of the results of single variable calculus carry over without much change.

Theorem 2. *Let* $\mathbf{f}, \mathbf{g}\colon [a,b] \to \mathbb{R}^2$ *be differentiable vector valued functions and define a scalar valued function by*

$$h(t) = \mathbf{f}(t) \cdot \mathbf{g}(t).$$

Then the product rule

$$h'(t) = \mathbf{f}'(t) \cdot \mathbf{g}(t) + \mathbf{f}(t) \cdot \mathbf{g}'(t)$$

holds. That is

$$\big(\mathbf{f}(t) \cdot \mathbf{g}(t)\big)' = \mathbf{f}'(t) \cdot \mathbf{g}(t) + \mathbf{f}(t) \cdot \mathbf{g}'(t).$$

Problem 5. Prove this. *Hint:* Reduce this to the one variable product as follows. Let $\mathbf{f}(t_=(f_1(t), f_2(t))$ and $\mathbf{g}(t) = (g_1(t), g_2(t))$. Then

$$\mathbf{f}(t)\mathbf{g}(t) = f_1(t)g_1(t) + f_2(t)g_2(t).$$

Then

$$\big(\mathbf{f}(t)\mathbf{g}(t)\big)' = \big(f_1(t)g_1(t)\big)' + \big(f_2(t)g_2(t)\big)'$$

You can now use the one variable product rule on the terms $\big(f_1(t)g_1(t)\big)'$ and $\big(f_1(t)g_2(t)\big)'$ and rearrange the results to get the desired formula. □

Corollary 3. *Let* $\mathbf{f}\colon [a,b] \to \mathbb{R}^2$ *be differentiable. Then*

$$\frac{d}{dt}\|\mathbf{f}(t)\|^2 = 2\mathbf{f}(t) \cdot \mathbf{f}'(t).$$

and at points where $\mathbf{f}(t) \neq \mathbf{0}$

$$\frac{d}{dt}\|\mathbf{f}(t)\| = \frac{\mathbf{f}(t)}{\|\mathbf{f}(t)\|} \cdot \mathbf{f}'(t).$$

Problem 6. Prove this. *Hint:* For the first formula let $\mathbf{g} = \mathbf{f}$ in Theorem 2. For the second use $\|\mathbf{f}(t)\| = \big(\|\mathbf{f}(t)\|^2\big)^{\frac{1}{2}}$. □

Here is anther product rule.

Proposition 4. *Let* $\mathbf{f}\colon [a,b] \to \mathbb{R}^2$ *be a differentiable vector valued function and let* $h\colon [a,b] \to \mathbb{R}$ *be a differentiable scalar valued function. Then*

$$\frac{d}{dt}\left(h(t)\mathbf{f}(t)\right) = h'(t)\mathbf{f}(t) + h(t)\mathbf{f}'(t).$$

Problem 7. Prove this. □

We now integrate vector valued functions. Let $\mathbf{f}\colon [a,b] \to \mathbb{R}^2$ be continuous and write it as

$$\mathbf{f}(t) = (f_1(t), f_2(t)).$$

Then

$$\int_a^b \mathbf{f}(t)\,dt = \left(\int_a^b f_1(t)\,dt, \int_a^b f_2(t)\,dt\right).$$

That is integrating a vector valued function is the same as integrating each of its component functions.

Proposition 5. *Let* $\mathbf{f}\colon [a,b] \to \mathbb{R}^2$ *be a vector valued function and* $\mathbf{a}$ *a constant vector. Then*

$$\mathbf{a} \cdot \int_a^b \mathbf{f}(t)\,dt = \int_a^b \mathbf{a} \cdot \mathbf{f}(t)\,t.$$

Problem 8. Prove this. *Hint:* Write $\mathbf{f}(t) = (f_1(t), f_2(t)$ and $\mathbf{a} = (a_1, a_2)$ and expand each side of the equation to be proven in terms of the definitions. □

Using that $\cos(\theta) \le 1$ for all θ we see that Equation (1) implies

$$\mathbf{a} \cdot \mathbf{b} \le \|\mathbf{a}\|\|\mathbf{b}\|$$

for all vectors $\mathbf{a}$ and $\mathbf{b}$. This is the ***Cauchy-Schwartz inequality***.

Theorem 6. *Let* $\mathbf{f}\colon [a,b] \to \mathbb{R}^2$ *be a continuous vector valued function. Then the inequality*

$$\left\| \int_a^b \mathbf{f}(t)\,dt \right\| \le \int_a^b \|\mathbf{f}(t)\|\,dt$$

holds.

Problem 9. If $\int_a^b \mathbf{f}(t)\,dt = \mathbf{0}$, then the result holds. So assume that $\int_a^b \mathbf{f}(t)\,dt \ne \mathbf{0}$. Now prove the result along the following lines.

(a) Let $\mathbf{a}$ be the vector

$$\mathbf{a} = \left\| \int_a^b \mathbf{f}(t)\,dt \right\|^{-1} \int_a^b \mathbf{f}(t)\,dt.$$

and show that $\mathbf{a}$ is a unit vector, that is

$$\|\mathbf{a}\| = 1$$

and that

$$\left\| \int_a^b \mathbf{f}(t)\,dt \right\| = \mathbf{a} \cdot \int_a^b \mathbf{f}(t)\,dt$$

(b) Now use Proposition 5 to show

$$\left\| \int_a^b \mathbf{f}(t)\,dt \right\| = \int_a^b \mathbf{a} \cdot \mathbf{f}(t)\,dt.$$

(c) Use that $\mathbf{a}$ is a unit vector and the Cauchy-Schwartz inequality to show

$$\mathbf{a} \cdot \mathbf{f}(t) \le \|\mathbf{f}(t)\|.$$

(d) Put all these pieces together to complete the proof. □