Mathematics 551 Homework, February 17, 2020

To make sure that my handwriting from class is accurately translated to the notation here, here is what the definition of some of main ideas. The notation for the tangent, normal, and binormal of a curve are $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ respectively. The curvature is κ and the torsion is τ. Thus the Frenet equations along a curve, γ, are

$$\begin{aligned} \frac{d\boldsymbol{\gamma}}{ds} &= \mathbf{t} \\ \frac{d\mathbf{t}}{ds} &= \qquad \kappa\mathbf{n} \\ \frac{d\mathbf{n}}{ds} &= -\kappa\mathbf{t} \qquad + \tau\mathbf{b} \\ \frac{d\mathbf{b}}{ds} &= \qquad -\tau\mathbf{n}. \end{aligned}$$

where s is arclength along the curve.

Problem 1 (*Path of a charged particle moving in a constant magnetic field*). The model used here for the motion of a particle moving in a constant magnetic field is that the force on the particle is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

where the scalar q is the charge on the particle, $\mathbf{v}$ is the velocity vector of the particle, the vector $\mathbf{B}$ is the magnetic field, and $\times$ is the usual cross product. If $\boldsymbol{\gamma}(t)$ is the position of the particle at time t then Newton's second law yields

$$m\frac{d^2\boldsymbol{\gamma}}{dt^2} = q\frac{d\boldsymbol{\gamma}}{dt} \times \mathbf{B}. \tag{1}$$

We will be assuming that $\mathbf{B}$ is constant. Our goal is to describe the motion of the particle.

(a) Show that the speed (that is the length of the velocity vector) of $\boldsymbol{\gamma}$ is constant. (Hint: for any vector $\mathbf{v}$ we have $\mathbf{v} \times \mathbf{B}$ is orthogonal to $\mathbf{v}$ and so by (1) $\frac{d^2\boldsymbol{\gamma}}{dt^2}$ and $\frac{d\boldsymbol{\gamma}}{dt}$ are orthogonal.) For the rest of this problem we let

$$v_0 = \left\| \frac{d\boldsymbol{\gamma}}{dt} \right\|. \tag{2}$$

be the speed of the particle.

(b) Let s be arclength along γ. Show that

$$\frac{d\boldsymbol{\gamma}}{dt} = v_0\frac{d\boldsymbol{\gamma}}{ds}, \qquad \frac{d^2\boldsymbol{\gamma}}{dt^2} = v_0^2\frac{d^2\boldsymbol{\gamma}}{ds^2}. \tag{3}$$

(c) Use ***(b)*** to show that Newton's second law (1) implies

$$\frac{d^2\boldsymbol{\gamma}}{ds^2} = \frac{d\boldsymbol{\gamma}}{ds} \times \mathbf{C} \tag{4}$$

where $\mathbf{C}$ is the constant vector

$$\mathbf{C} = \left(\frac{q}{mv_0}\right)\mathbf{B}. \tag{5}$$

(d) Take derivatives of (3) with respect to s to show

$$\frac{d^3\boldsymbol{\gamma}}{ds^3} = \frac{d^2\boldsymbol{\gamma}}{ds^2} \times \mathbf{C}, \qquad \frac{d^4\boldsymbol{\gamma}}{ds^4} = \frac{d^3\boldsymbol{\gamma}}{ds^3} \times \mathbf{C} \tag{6}$$

and therefore

$$\left\|\frac{d^2\boldsymbol{\gamma}}{ds^2}\right\| \quad \text{and} \quad \left\|\frac{d^3\boldsymbol{\gamma}}{ds^3}\right\| \tag{7}$$

are constant.

(e) Let θ be the angle between $\frac{d\boldsymbol{\gamma}}{ds}$ and $\mathbf{C}$ (which is the same as the angle between $\frac{d\boldsymbol{\gamma}}{dt}$ and $\mathbf{B}$). Show that θ, the curvature, κ, and the torsion, τ, are all constant along $\boldsymbol{\gamma}$. (This means that the motion of the particle is a helix (or a circle if $\tau = 0$) and the of the axis of the helix is parallel to the direction of $\mathbf{B}$.)