Mathematics 551 Homework, March 7, 2020

Problem 1. In this problem you will derive some standard formulas for the first and second fundamental forms of graphs. Let $U \subseteq \mathbb{R}^2$ be an open set and let $f\colon U \to \mathbb{R}$ be a smooth function. Define a $\mathbf{x}\colon U \to \mathbb{R}^3$ by

$$\mathbf{x}(u,v) = (u, v, f(u,v)).$$

Let M be the surface parameterized by this function $\mathbf{x}$. That is M is the graph of the function $z = f(x,y)$.

(a) Show

$$\mathbf{x}_u = (1, 0, f_u)$$
$$\mathbf{x}_v = (0, 1, f_v)$$

(b) Show that the first fundamental form is

$$I = (1 + f_u^2)\, du^2 + 2 f_u f_v\, dudv + (1 + f_v^2)\, dv^2.$$

(c) Show the unit normal is

$$\mathbf{n}(u,v) = (1 + f_u^2 + f_v^2)^{-1/2}(-f_u, -f_v, 1).$$

(d) Find the second fundamental form of $\mathbf{x}$. $\square$

Problem 2. In the last problem let us consider the special case were

$$f(0,0) = f_u(0,0) = f_v(0,0) = 0$$

and let M be the surface which is the graph of $z = f(x,y)$. Then the graph will be tangent to the x-y plane at the origin. Assume

$$f_{uu}(0,0) = k_1$$
$$f_{uv}(0,0) = 0$$
$$f_{vv}(0,0) = k_2$$

where k_1 and k_2 are constants. As in the previous problem let

$$\mathbf{x}(u,v) = (u, v, f(u,v)).$$

(a) Show that the first and second fundamental forms of $\mathbf{x}$ at the origin are

$$I_{(0,0,0)} = du^2 + dv^2$$
$$II_{(0,0,0)} = k_1\, du^2 + k_2\, dv^2$$

and that the normal at the origin is

$$\mathbf{n}(0,0) = (0,0,1).$$

(This should follow at once from Problem 1.)

(b) Show that at the origin the shape operator $S = S_{(0,0,0)}$ satisfies satisfies

$$S\mathbf{x}_u(0,0) = k_1 \mathbf{x}(u,v), \qquad S\mathbf{x}_v(0,0) = k_2 \mathbf{x}_v(0,0).$$

Thus k_1 and k_2 are the eigenvalues of S.

(c) One way to understand how a surface is curved is to intersect it with planes and look at the curvature of the resulting curve. Let us look at an example of this. Let $\mathcal{P}_\theta$ be the plane spanned by

$$E_1(\theta) = (\cos(\theta), \sin(\theta), 0), \qquad E_3 = (0,0,1) = \mathbf{n}(0,0).$$

Show that the curve of intersection $\mathcal{P}_\theta \cap M$ is parameterized by

$$\begin{aligned}\boldsymbol{\gamma}(t) &= (t\cos(\theta), t\sin(\theta), f(t\cos(\theta), t\sin(\theta))) \\ &= tE_1(\theta) + f(t\cos(\theta), t\sin(\theta))E_3.\end{aligned}$$

Show that the curvature of this curve (viewed as a curve in $\mathcal{P}_\theta$) at the origin is

$$\kappa(0) = k_1\cos^2(\theta) + k_2\sin^2(\theta).$$

This is theorem due to Euler. □

Now let us look at the example of a sphere of radius R centered at the origin. Call this sphere $S^2(R)$. Let $U \subseteq \mathbb{R}^2$ be an open set and $\mathbf{x}\colon U \to S^2(R)$ be a local parameterization of $S^2(R)$. Then

$$\mathbf{x}(u,v)\cdot\mathbf{x}(u,v) = R^2$$

for all $(u,v) \in U$. Taking the derivatives gives

$$2\mathbf{x}_u \cdot \mathbf{x} = 0, \qquad 2\mathbf{x}_v \cdot \mathbf{x} = 0.$$

Therefore $\mathbf{x}$ is a normal to the surface and so the unit normal will be one of

$$\frac{1}{R}\mathbf{x} \qquad \text{or} \qquad -\frac{1}{R}\mathbf{x}.$$

We assume that our set up is so that

$$\mathbf{n}(u,v) = -\frac{1}{R}\mathbf{x}.$$

Problem 3. Recall that the shape operator, S, is the linear map on tangent spaces to the surface defined by

$$S(\mathbf{x}_u) = -D_{x_u}\mathbf{n}(u,v), \qquad S(\mathbf{x}_v) = -D_{\mathbf{x}_v}\mathbf{n}(u,v).$$

Or to be a bit more concrete (by unscrambling the definition of the directional derivative $D_{\mathbf{x}_u}$) this is the same as

$$S(\mathbf{x}_u) = -\frac{\partial\mathbf{n}}{\partial u}, \qquad S(\mathbf{x}_v) = -\frac{\partial\mathbf{n}}{\partial v}.$$

Since S is linear knowing what it does to the basis is enough determine what it does to arbitrary vectors. Now back to the example of the sphere of radius R centered at the origin where

$$\mathbf{n} = -\frac{1}{R}\mathbf{x}.$$

Show that in this case the shape operator satisfies

$$S(\mathbf{x}_u) = \frac{1}{R}\mathbf{x}_u, \qquad S(\mathbf{x}_v) = \frac{1}{R}\mathbf{x}_v$$

and therefore on the sphere the shape operator is given by

$$S = \frac{1}{R} I$$

where I is the identity map. □

Thus you have proven:

Proposition 1. *On the sphere of radius R the shape operator with respect to the inward unit normal is $S = (1/R)I$ where I is the identity map.* □

This has a converse:

Theorem 2. *Let $\mathbf{x}\colon U \to \mathbb{R}^3$ be a C^2 map on the connected open subset U of $\mathbb{R}^2$. Assume that the shape operator of $\mathbf{x}$ is*

$$S = kI$$

where $k \neq 0$ is a constant. Then $\mathbf{x}$ lies in a sphere of radius $R = 1/|k|$.

Problem 4. Prove this. *Hint:* Let $\mathbf{c}(u, v)$ be

$$\mathbf{c}(u, v) = \mathbf{x}(u, v) + \frac{1}{k}\mathbf{n}(u, v).$$

Use that $S(\mathbf{x}_u) = \frac{1}{k}\mathbf{x}_u$ and $S(\mathbf{x}_v) = \frac{1}{k}\mathbf{x}_v$ to show $\mathbf{c}_u = \mathbf{c}_u = \mathbf{0}$ and therefore $\mathbf{c}$ is constant. □

Problem 5. Let $S^2(R)$ be the sphere of radius R and let U be an open subset of $\mathbb{R}^2$. If R is the radius of the earth explain why a cartographer would like to find a map $\mathbf{x}\colon U \to S^2(R)$ and a constant c such that the first fundamental from of $\mathbf{x}$ is

$$I = c^2(du^2 + dv^2).$$

Hint: Think of U as a map where each point (u, v) corresponds to the point $\mathbf{x}(u, v)$ on $S^2(R)$, which we think of as the surface of the earth. Let $\boldsymbol{\alpha} \to [a, b]\colon U$ be a C^1 curve on the map such that $\boldsymbol{\gamma}(t) = \mathbf{x}(\boldsymbol{\alpha}(t))$ moves over a road, say I-26 between Columbia and Charleston. If $I = c^2(du^2 + dv^2)$ show

$$\operatorname{Length}(\boldsymbol{\gamma}) = c \operatorname{Length}(\boldsymbol{\alpha}).$$

What does this say about the relation of distances on the map and corresponding distance on the surface of the earth? □

Problem 6 (Optional). Show that it is impossible to find a map $\mathbf{x}\colon U \to S^2(R)$ as in the previous problem. We will soon show a general result that shows this, but it is interesting to try doing it with bare hands. *Hint:* The idea is to start taking derivatives with respect to u and v of $\mathbf{x} \cdot \mathbf{x} = R^2$ until you get a contradiction. □

Problem 7. Let $\mathbf{x}\colon U \to \mathbb{R}^3$ be a C^2 where U is an open set in $\mathbb{R}^2$ such that the first fundamental form of $\mathbf{x}$ is

$$I = \rho(u, v)^2(du^2 + dv^2)$$

for some function ρ. Show that $\mathbf{x}$ preserves angles. More explicitly this means that if $\mathbf{c}_1, \mathbf{c}_2(-\delta, \delta) \to U$ are curves with $\mathbf{c}_1(0) = \mathbf{c}_2(0)$ (that is they both go through the same point at time $t = 0$) and if $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \colon (-\delta, \delta) \to \mathbb{R}^3$ are the curves

$$\boldsymbol{\gamma}_j(t) = \mathbf{x}(\mathbf{c}_j(t)) \qquad \text{for} \qquad j = 1, 2.$$

then

$$\sphericalangle(\boldsymbol{\gamma}_1'(0), \boldsymbol{\gamma}_2'(0)) = \sphericalangle(\mathbf{c}_1'(0), \mathbf{c}_2'(0)). \qquad \square$$