The curvature of level sets of smooth functions of two variables.

We know how to compute the curvature of a parametric curve $\boldsymbol{\gamma}\colon (\alpha,\beta)\to \mathbf{R}^2$. But sometimes curves are given as level sets of a function, that is as the set of points where $f(x,y)=c$ where c is a constant. It is then useful to be able to compute the curvature directly without having to first find a parameterization of the curve. In this problem set we will be using the notation common in physics, engineering, and parts of geometry where derivatives are denoted by using dots rather than primes. That is if $x = x(t)$, then the derivative will be written as $\dot{x}$ rather than x' and the second derivative is $\ddot{x}$. (I believe that one reason this is popular in the physical sciences is that it was Newton's notation for the derivative.)

Let $f(x,y)$ be a smooth function and c a constant such that if $f(x,y)=c$, then $\nabla f(x,y)\neq(0,0)$, where $\nabla f=(f_x,f_y)$ is the gradient of f. Let $\mathcal{C}$ be the level set

$$\mathcal{C} := \{(x,y) : f(x,y)=c\}$$

There is a theorem from advanced calculus, *The Implicit Function Theorem*, which tells us that near any of its points $\mathcal{C}$ has a parameterization as regular curve. It then has parameterization as a unit speed curve. Let $\boldsymbol{\gamma}\colon(\alpha,\beta)\to\mathcal{C}$ be a unit speed parameterization of $\mathcal{C}$. (In what follows we do not need to find $\boldsymbol{\gamma}$ explicitly, it is enough to know that it exists.) From the definition of $\mathcal{C}$

$$f(\boldsymbol{\gamma}(s)) = c.$$

Taking the derivative and using chain rule gives

$$\frac{d}{ds}f(\boldsymbol{\gamma}(s)) = \nabla f(\boldsymbol{\gamma}(s))\cdot\dot{\boldsymbol{\gamma}}(s) = f_x(x(s),y(s))\dot{x}(s) + f_y(x(s),y(s))\dot{y}(s) = 0$$

and therefore $\dot{\boldsymbol{\gamma}}(s)$ is perpendicular to $\nabla f(\boldsymbol{\gamma}(s))$.

We still have to decide which direction we are traveling along $\mathcal{C}$. Our convention will be that we move along $\mathcal{C}$ so that $\mathbf{n}(s)$ points in the direction of $-\nabla f$. That is we move moving along $\boldsymbol{\gamma}(s)$ keeping the down hill side on our left.

Problem 1. If $f(x,y)=x^2+y^2$ and $r>0$, then the level set $\mathcal{C}=\{(x,y): x^2+y^2=r^2\}$ is the circle of radius r. Draw a picture and use it to explain[1] why this convention gives that $\boldsymbol{\gamma}(s)$ is moving along $\mathcal{C}$ in the positive (that is counterclockwise) direction. □

Let $\boldsymbol{\gamma}(s)$ be given by

$$\boldsymbol{\gamma}(s) = (x(s),y(s)).$$

As $\boldsymbol{\gamma}(s)$ is unit speed this implies

$$\dot{\boldsymbol{\gamma}}(s) = (\dot{x}(s),\dot{y}(s)) = \mathbf{t}(s), \qquad \ddot{\boldsymbol{\gamma}}(s) = (\ddot{x}(s),\ddot{y}(s)) = \kappa(s)\mathbf{n}(s)$$

[1]This means that besides the picture there should be at least one, preferably more, explaining why the picture is relevant.

where $\mathbf{t}(s)$ and $\mathbf{n}(s)$ are the unit tangent and unit normal along $\boldsymbol{\gamma}$ and $\kappa(s)$ is the curvature. Anther way of stating our convention on the direction we are moving along $\mathcal{C}$ is that

$$\mathbf{n}(s) = (-\dot{y}(s), \dot{x}(s)) = \frac{-1}{\|\nabla\|}(f_x, f_y) = \frac{1}{\sqrt{f_x^2 + f_y^2}}(-f_x, -f_y)$$

Problem 2. Use these formulas to show

$$\mathbf{t}(s) = \frac{1}{\|\nabla f\|}(-f_y, f_x)$$

along $\mathcal{C}$ and therefore the formulas

$$\dot{x} = \frac{-f_y}{\|\nabla f\|}, \quad \dot{y} = \frac{f_x}{\|\nabla f\|}, \qquad \ddot{x} = \frac{-\kappa f_x}{\|\nabla f\|}, \quad \ddot{y} = \frac{-\kappa f_y}{\|\nabla f\|}$$

hold. □

Problem 3. Explain way the formula

$$f_{xx}\dot{x}^2 + 2f_{xy}\dot{x}\dot{y} + f_{yy}\dot{y}^2 + f_x\ddot{x} + f_y\ddot{y} = 0$$

holds along $\mathcal{C}$.(HINT: Take two derivatives of $f(x, y) = c$.) □

Problem 4. Use the formulas from Problem 2 in the formula from Problem 3 to find a formula for the curvature, κ, of $\mathcal{C}$, in terms of f_x, f_y, f_{xx}, f_{xy} and f_{yy}. (Note our finial formula does not involve the parameterization $\boldsymbol{\gamma}(s)$, we just needed it in the intermediate steps.) □

Problem 5. As a check to your solution to the last problem, use it to compute the curvature of $f(x, y) = x^2 + y^2 = r^2$ (you should get $\kappa = 1/r$) and $f(x, y) = g(x) - y = 0$ (you should get $\kappa = g''(x)/(1 + g'(x)^2)^{3/2}$). □