Mathematics 552 Homework, January 21, 2020

Infinite series play a big part in the theory of complex variable. In this set of notes/homework we recall some basic facts, mostly about the special case of geometric series.

1. Summation notation.

Summation notation will be used a great deal in this class. We recall the basics about it. The notation is

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_{n-1} + a_n.$$

Thus

$$\sum_{k=0}^{5} ar^k = a + ar + ar^2 + ar^3 + ar^4 + ar^5.$$

There is nothing special about using k for the index:

$$\sum_{k=1}^{100} a_k = \sum_{j=1}^{100} a_j = \sum_{\alpha=1}^{100} a_\alpha = \sum_{\smiley=1}^{100} a_{\smiley} = \sum_{\frownie=1}^{100} a_{\frownie}.$$

A basic property of sums is

$$c_1 \sum_{k=m}^{n} a_k + c_2 \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (c_1 a_k + c_2 b_k).$$

We will also want to do changes of index in sum. For example

$$\begin{aligned} \sum_{k=m}^{n} a_k x^{k+3} &= a_m x^{m+3} + a_{m+1} xm + 4 + \cdots + a_{n-1} x^{n+2} + a_n x^{n+3} \\ &= \sum_{k=m+3}^{n+3} a_{k-3} x^k. \end{aligned}$$

And of course summation notation also works for infinite series. Thus

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

when this series converges.

2. Geometric series.

2.1. **Finite geometric series.** A (finite) ***geometric series*** is a finite sum of the form

$$S = a + ar + ar^2 + \cdots + ar^n.$$

In summation notation this is

$$S = \sum_{k=0}^{n} ar^k.$$

Such sums occur naturally in many contexts and fortunately it is easy give a formula for their sum. We first look at the case of $n = 2$. Then

$$S = a + ar + ar^2.$$

Multiply this by r to get

$$rS = ar + ar^2 + ar^3.$$

Note that the sums for S and rS have the terms ar and ar^2 in common, which suggests subtracting to cancel these terms out:

$$\begin{aligned} S &= a + ar + ar^2 \\ -rS &= \quad\, - ar - ar^2 - ar^3 \\ S - rS &= a - ar^3. \end{aligned}$$

Therefore

$$(1 - r)S = a - ar^3$$

which, when $r \neq 1$, we can solve for S to get

$$S = \frac{a - ar^3}{1 - r}$$

For $n = 5$ the calculation looks like

$$\begin{aligned} S &= a + ar + ar^2 + ar^3 + ar^4 + ar^5 \\ -rS &= \quad\, - ar - ar^2 - ar^3 - ar^4 - ar^5 - ar^6 \\ S - rS &= a - ar^6 \end{aligned}$$

and therefore

$$(1 - r)S = a - ar^6.$$

So when $r \neq 1$ we have

$$S = \frac{a - ar^6}{1 - r}.$$

At this point you have likely guessed the general pattern:

Theorem 1. *Let a and r be real numbers with $r \neq 1$ and $n \geq 2$ and integer. Then the sum of the geometric series*

$$S = a + ar + ar^2 + \cdots + ar^n$$

is

$$S = \frac{a - ar^{n+1}}{1 - r}.$$

Problem 1. Prove this. □

Problem 2. What happens in the theorem when $r = 1$? □

The way I find easiest to remember and apply this is to note that if the series $a + ar + ar^2 + \cdots + ar^n$ is continued that the next term would be ar^{n+1}. Therefore if we call the number r the ***ratio*** then

$$a + ar + ar^2 + \cdots + ar^n = \frac{1 - \text{next term}}{1 - \text{ratio}}.$$

Here are some examples

$$x^2 + x^4 + x^6 + \cdots + x^{20} = \frac{\text{first} - \text{next term}}{1 - \text{ratio}} = \frac{x^2 - x^{22}}{1 - x^2}$$

holds when $x \neq \pm 1$.

Let

$$S = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64}.$$

Then

$$\begin{aligned} S &= \frac{1 - \text{next term}}{1 - \text{ratio}} \\ &= \frac{1 - (-1/128)}{1 - (-1/2)} = \frac{128 + 1}{128 + 64} = \frac{129}{192}. \end{aligned}$$

Let

$$\alpha = \overbrace{.333 \cdots 3}^{n \text{ digits}}.$$

Then

$$\begin{aligned} \alpha &= 3(.1) + 3(.1)^2 + 3(.1)^3 + \cdots + 3(.1)^n \\ &= \frac{\text{first} - \text{next}}{1 - \text{ratio}} \\ &= \frac{3(.1) - 3(.1)^{n+1}}{1 - .1} \\ &= \frac{.3 - .3(.1)^n}{.3(3)} \\ &= \frac{1}{3} - \frac{1}{3(10)^n} \end{aligned}$$

There is anther natural way to find α:

$$\begin{aligned} 9\alpha = 10\alpha - \alpha &= (3.33 \cdots 3) - (.333 \cdots 3) \\ &= 3 - \underbrace{.000 \cdots 3}_{10 \text{ decmal places}} \end{aligned}$$

Therefore

$$\alpha = \frac{3 - .000 \cdots 3}{9} = \frac{1}{3} - \frac{.000 \cdots 1}{3} = \frac{1}{3} - \frac{1}{3(10)^n}$$

For the classical problem[1] of putting one grain rice on the first square of a chess broad, two on the second square, four on the third square, eight on the fourth square: that is doubling the number on each square up until the 64th square, then the total number of grains is

$$1+2+4+\cdots+2^{63}=\frac{1-2^{64}}{1-2}=2^{64}-1=18{,}446{,}744{,}073{,}709{,}551{,}615.$$

Remark 2. The internet tells me that "A single long grain of rice weighs an average of 0.001 ounces (29 mg)." Thus the total weight of the rice on the chess board is $(2^{64}-1)/(1{,}000)$ onces. The number of onces $(2^{64}-1)/(1{,}000)$ in a ton is $2{,}000\times 16=32{,}000$. Therefore the weight in tons of the rice

$$W=(2^{64}-1)/(1{,}000\times 32{,}000)=5.76460752303423\times 10^{11}.$$

The internet also says that the current rate of world rice production is about $P=7.385477\times 10^{8}$ tones/year. At this rate is would take about

$$\frac{W}{P}\approx 780.533$$

years to cover the chess board. □

Problem 3. (a) Find the sum of $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}$

(b) Find the sum of $P_0(1+r)+P_0(1+r)^2+\cdots+P_0(1+r)^{n-1}$. (If at the beginning of each year you put P_0 in a bank account that pays simple interest at a rate of $100r\%$ per year, then this sum is the total after n years. Thus there are a total of $n-1$ depostes. As a check on your answer when $P_0=1{,}000$ and $r=.05$, (that is a 5% simple interest) then after 20 years the total is, to the nearest penny, 33065.95.) □

2.2. **Infinite geometric series.** If $|r|<1$, then r^n gets closer and closer to 0 as n gets larger. That is

$$lim_{n\to\infty} r^n=0.$$

Therefore if $|r|<1$ we can take the limit as $n\to\infty$ in the formula for the sum of a finite geometric series:

$$\sum_{k=0}^{n} ar^k=a+ar+ar^2+\cdots+ar^n$$

[1] From the Wikipedia article on putting on grains of rice (or wheat) *Wheat and chessboard problem* `https://en.wikipedia.org/wiki/Wheat_and_chessboard_problem` The problem appears in different stories about the invention of chess. One of them includes the geometric progression problem. The story is first known to have been recorded in 1256 by Ibn Khallikan.[1] Another version has the inventor of chess (in some tellings Sessa, an ancient Indian Minister) request his ruler give him wheat according to the wheat and chessboard problem. The ruler laughs it off as a meager prize for a brilliant invention, only to have court treasurers report the unexpectedly huge number of wheat grains would outstrip the ruler's resources. Versions differ as to whether the inventor becomes a high-ranking advisor or is executed.

to find the sum of an infinite geometric series with whose ratio r satisfies $|r| < 1$ to get

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r} = \frac{\text{first}}{1-\text{ratio}}.$$

As an example let $|x| < 1$ then

$$2x - 2x^3 + 2x^5 - 2x^7 + \cdots = \sum_{k=0}^{\infty} 2(-1)^k x^{2k+1}$$

is a geometric series with first term $2x$ and ratio $-x^2$. Therefore its sum is

$$2x - 2x^3 + 2x^5 - 2x^7 + \cdots = \frac{\text{first}}{1-\text{ratio}} = \frac{2x}{1-(-x^2)} = \frac{2x}{1+x^2}.$$

Problem 4. If $|x| < 1$ find the sum of the following geometric series:

(a) $\displaystyle\sum_{j=1}^{\infty} \frac{1}{2^j}$

(b) $\displaystyle\sum_{k=0}^{\infty} (-1)^k x^{3k+1}$ $\square$

3. Some useful factoring formulas.

You recall that

$$x^2 - y^2 = (x-y)(x+y)$$

and may recall that

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2).$$

These generalize. To see how let us look at the right hand side of the last equation. If we multiple this out we get

$$\begin{aligned}(x-y)(x^2+xy+y^2) &= x(x^2+xy+y^2) - y(x^2+xy+y^2)\\ &= x^3 + x^2y + xy^2 - x^2y - xy^2 - y^3 \qquad \text{(most terms cancel)}\\ &= x^3 - y^3.\end{aligned}$$

Let us look at a similar product:

$$\begin{aligned}(x-y)(x^3+x^2y+xy^2+y^3) &= x(x^3+x^2y+xy^2+y^3) - y(x^3+x^2y+xy^2+y^3)\\ &= x^4 + x^3y + x^2y^2 + xy^3\\ &\qquad - x^3y - x^2y^2 - xy^3 - y^4\\ &= x^4 - y^4.\end{aligned}$$

And just to be sure we see the pattern let us look at the next case

$$\begin{aligned}(x-y)(x^4+x^3y+x^2y^2+xy^3+y^4) &= x(x^4+x^3y+x^2y^2+xy^3+y^4)\\ &\quad - y(x^4+x^3y+x^2y^2+xy^3+y^4)\\ &= x^5+x^4y+x^3y^2+x^2y^3+xy^4\\ &\quad - x^4y-x^3y^2-x^2y^3-xy^4-y^5\\ &= x^5-y^5.\end{aligned}$$

The pattern is now clear:

Theorem 3. *Let n be any positive integer and let x and y be any two real numbers. Then*

$$x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+xy^{n-2}+y^{n-1}).$$

In summation notation this is

$$x^n-y^n=(x-y)\left(\sum_{k=0}^{n-1}x^{n-1-k}y^k\right)=(x-y)\left(\sum_{\substack{j+k=n-1\\0\le j,k\le n-1}}x^jy^k\right)$$

Problem 5. Prove this by multiplying out $(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+xy^{n-2}+y^{n-1})$ and seeing that all but two terms cancel. □

Problem 6. The proof of Theorem 3 may remind you of the proof of Theorem 1 because both rely on a lot of cancellation. This is because there is a geometric series hidden in the proof of Theorem 3. Let use consider the case of $n=5$ and set

$$S=x^4+x^3y+x^2y^2+xy^3+y^4.$$

This can be written as

$$S=x^4+x^4\left(\frac{y}{x}\right)+x^4\left(\frac{y}{x}\right)^2+x^4\left(\frac{y}{x}\right)^3+x^4\left(\frac{y}{x}\right)^4$$

which is a geometric series. Thus

$$\begin{aligned}S&=\frac{\text{first}-\text{next}}{1-\text{ratio}}\\ &=\frac{x^4-x^4\left(\frac{y}{x}\right)^5}{1-\frac{y}{x}}\\ &=\frac{x^5-y^5}{x-y}.\end{aligned}$$

Recalling the definition of S this is

$$x^4+x^3y+x^2y^2+xy^3+y^4=\frac{x^5-y^5}{x-y}$$

which is equivalent to the $n=5$ version of Theorem 3. Give a proof of general case of Theorem 3 using the method just given. □