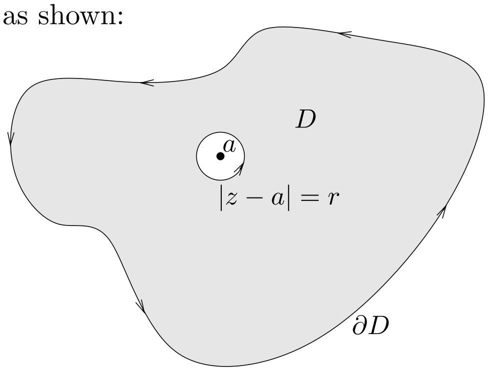
Mathematics 552 Homework, March 4, 2020

There will be a quiz on Friday where you will be ask to prove the Cauchy integral formula along the following lines.

Let $f(z)$ be analytic on a bounded open set D and its boundary ∂D and assume that ∂D is nice enough that Green's Theorem applies. Let $a \in D$ and let $r > 0$ be small enough that the disk

$$B(a,r) = \{z : |z-a| < r\}$$

is contained in D as shown:



Problem 1. Explain why

$$\int_{\partial(D \setminus B(a,r))} \frac{f(z)}{z-a}\,dz = 0.$$

Solution: The function $g(z) := \dfrac{f(z)}{z-a}$ is analytic at all points of D other than $z = a$. But a is not in $D \setminus B(a,r)$ and so $g(z)$ is analytic in the domain $D \setminus B(a,r)$ and therefore by the Cauchy Integral Theorem $\int_{D \setminus B(a,r)} g(z)\,dz = 0$. $\square$

Problem 2. Explain why

$$\int_{\partial D} \frac{f(z)}{z-a}\,dz = \int_{|z-a|=r} \frac{f(z)}{z-a}\,dz.$$

Solution: From the last problem we have

$$\begin{aligned} 0 &= \int_{\partial(D \setminus B(a,r))} \frac{f(z)}{z-a}\,dz \\ &= \int_{\partial D} \frac{f(z)}{z-a}\,dz - \int_{\partial B(a,r)} \frac{f(z)}{z-a}\,dz \\ &= \int_{\partial D} \frac{f(z)}{z-a}\,dz - \int_{|z-a|=r} \frac{f(z)}{z-a}\,dz. \end{aligned}$$

This implies

$$\int_{\partial D} \frac{f(z)}{z-a}\,dz = \int_{|z-a|=r} \frac{f(z)}{z-a}\,dz$$

as required. $\square$

Problem 3. Use the parameterization $z = a + re^{it}$ with $0 \le t \le 2\pi$ of the circle $|z-a| = r$ to show

$$\int_{|z-a|=r} \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a+re^{it})\,dt.$$

Solution: If $z = a + re^{it}$, then $dz = ire^{it}\,dt$. Therefore

$$\begin{aligned}\int_{|z-a|=r} \frac{f(z)}{z-a}\,dz &= \int_0^{2\pi} \frac{f(a+re^{it})}{(a+re^{it})-a}(ire^{it}\,dt)\\ &= \int_0^{2\pi} \frac{f(a+re^{it})}{re^{it}}\,ire^{it}\,dt\\ &= i\int_0^{2\pi} f(a+re^{it})\,dt.\end{aligned}$$

$\square$

Problem 4. Show

$$\lim_{r\to 0^+}\int_{|z-a|=r} \frac{f(z)}{z-a}\,dz = 2\pi i f(a).$$

Solution: We use Problem 3:

$$\begin{aligned}\lim_{r\to 0^+}\int_{|z-a|=r} \frac{f(z)}{z-a}\,dz &= \lim_{r\to 0} i\int_0^{2\pi} f(a+re^{it})\,dt\\ &= i\int_0^{2\pi} \lim_{r\to 0^+} f(a+re^{it})\,dt\\ &= i\int_0^{2\pi} f(a+0)\,dt\\ &= i\int_0^{2\pi} f(a)\,dt\\ &= if(a)t\Big|_0^{2\pi}\\ &= 2\pi i f(a).\end{aligned}$$

$\square$

Problem 5. Prove

$$f(a) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{z-a}\,dz.$$

Solution: This is just a matter of putting the Problems above together.

$$\begin{aligned}\int_{\partial D} \frac{f(z)}{z-a}\,dz &= \int_{|z-a|=r} \frac{f(z)}{z-a}\,dz && \text{(by Problem 2)}\\ &= \lim_{r\to 0^+}\int_{r\to 0^+} \frac{f(z)}{z-a}\,dz && \text{(as this is constant as function of } r\text{)}\\ &= 2\pi i f(a) && \text{(by Problem 4).}\end{aligned}$$

Dividing by $2\pi i$ finishes the proof. $\square$