Mathematics 552 Homework, January 29, 2020

The book takes a different approach to defining the exponential function e^z. We defined it in terms of the series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

The book defines it (see Equation (2.4) in Section 2.6) as

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y),$$

which we proved as consequences of ***Euler's formula***

$$e^z = \cos(z) + i\sin(z)$$

and that $e^{z+w} = e^z e^w$, where we have defined $\sin(z)$ and $\cos(z)$ by their series

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \cdots$$
$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \cdots$$

We showed in class that Euler's formula implies

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$
$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

The book uses these for the definitions of $\cos(z)$ and $\sin(z)$.

We say that w is a ***logarithm*** of z if and only if

$$e^w = z.$$

Note that a complex number generally has infinitely many logarithms. For example we have seen that the general solution to

$$e^w = 1$$

is

$$w = 2n\pi i \quad \text{where} \quad n \in \mathbb{Z}.$$

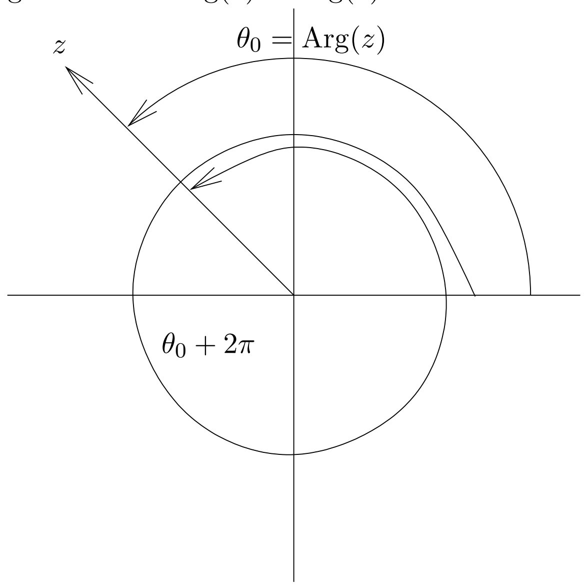
Therefore $\ln(1) = 2n\pi i$ where $n \in \mathbb{Z}$.

The logarithms of a complex number are closely related to its polar form. Let

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

with $r > 0$ and $\theta \in \mathbb{R}$. The angle θ is call an ***argument*** of z and denoted by $\arg(z)$. (See Section 1.8 on Page 4 of the text for more about this.) There are infinitely many choices for $\theta = \arg(z)$. But any two of them differ by an integer number of whole revolutions, that is by an integer multiple of $2\pi = 360°$. The ***principle value*** of the argument (see Section 1.8 on

Page 4 of the text) is the argument θ with $-\pi < \theta \leq \pi$. We will denote the principle argument by $\mathrm{Arg}(z)$. The figure shows the principle value of z along anther argument with $\arg(z) = \mathrm{Arg}(z) + 2\pi$.



You should now look at Problem 2.13 on Page 56 of the text along with its solution. There it is shown that the general solution to

$$e^w = z$$

that is the general form of the logarithm of z is

$$\ln z = \ln|z| + i \arg z.$$

Put somewhat differently $w = \ln|z| + i \arg z$ is the general solution to $e^w = z$. Note that there are infinitely many solutions as there are infinitely many choices for $\arg z$. To be more explicit we can write

$$\ln z = \ln|z| + i\,\mathrm{Arg}\, z + 2n\pi i$$

where $n \in \mathbb{Z}$.

Problem 1. Part (b) of Problem is to find the values of $\ln(1 - i)$. Use the method of that problem to find

(a) all values of $\ln(2 - 2\sqrt{3}i)$
(b) all solutions to $e^{2z} + 2e^z + 2 = 0$. *Hint:* If $w = e^z$ this equation becomes the quadratic equation $w^2 + 2w + 2 = 0$. Solving this gives two solutions: $w = -1 + i$ and $w = -1 - i$. So the original problem now splits into the two problems of finding all solutions to $e^z = -1 + i$ and $e^z = -1 - i$.
(c) Find all solutions to $e^{2z} - e^z - 2 = 0$. □

Problem 2. Find all solutions to $\sin(z) = 2$.

We have defined two new functions:

$$\cosh(z) = \frac{e^z + e^{-z}}{2}$$
$$\sinh(z) = \frac{e^z - e^{-z}}{2}$$

Problem 3. Show these functions satisfy

$$\cosh^2(z) - \sinh^2(z) = 1. \qquad \square$$

Problem 4. Show that as functions of the real variable x that the derivatives of $\cosh(x)$ and $\sinh(x)$ are given by

$$\frac{d}{dx}\cosh(x) = \sinh(x)$$
$$\frac{d}{dx}\sinh(x) = \cosh(x)$$

One reason we are interested in these functions is that we will shortly want to find the real and imaginary parts of $\cos(z) = \cos(x+iy)$ and $\sin(z) = \sin(x+iy)$.

Proposition 1. *The formulas*

$$\cos(x+iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$
$$\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$

hold.

Problem 5. Prove this. *Hint:* One way is to use the definitions and Euler's formulas. The first couple of steps in the case of $\cos(x+iy)$ then looks like

$$\begin{aligned}
\cos(x+iy) &= \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} \\
&= \frac{e^{-y+ix} + e^{y-ix}}{2} \\
&= \frac{e^{-y}e^{ix} + e^{y}e^{-ix}}{2} \\
&= \frac{e^{-y}(\cos(x) + i\sin(x)) + e^{y}(\cos(x) - i\sin(x))}{2}
\end{aligned}$$

and now split this into its real and imaginary parts. A similar calculation works for $\sin(x+iy)$.