Mathematics 552 Homework, February 5, 2020

Problem 1. Find a general formula for the solutions to $\sin(w) = z$ and $\cos(w) = z$. That is solve $\sin(w) = z$ for w (and likewise solve $\cos(w) = z$ for w). *Hint:* If you are having trouble look at the text for the formulas for $\arcsin(z)$ and $\arccos(z)$.

Here we will mostly be looking at consequences of the Cauchy-Riemann equations. That is if $f = u+iv$ is analytic in an open set, then the equations

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x.$$

hold in that set.

Definition 1. Let U be an open set in the complex plane $\mathbb{C}$. Then a function $h\colon U \to \mathbb{R}$ is ***harmonic*** iff

$$h_{xx} + h_{yy} = 0.$$

(It is being assumed that the first and second partial derivatives of h exist and are continuous.) $\square$

A very important result is that the real and imaginary parts of an analytic function are harmonic. To be precise

Theorem 2. *Let $f = u + iv$ be analytic in the open set U. Assume that u and v have continuous first and second partial derivatives. Then both u and v are harmonic.*

While this is important it is not hard:

Problem 2. Prove the last theorem. *Hint:* It is a more or less direct consequence of the Cauchy-Riemann equations. As a start note

$$u_{xx} = (u_x)_x = (v_y)_x = v_{xy}$$

with a similar formula for u_{yy} in terms of v_{xy}. If you want more of a hint see Problem 3.6 in the text. $\square$

A consequence of our proof of the Cauchy-Riemann equations is

Proposition 3. *Let $f = u + iv$ be analytic in an open set U. Then the derivative of f is give by either of the formulas*

$$f' = u_x + iv_x \qquad \text{and} \qquad f' = v_y - iu_y$$

(In practice we usually just use $f' = u_x + iv_x$.) $\square$

Here is an example similar to an example we did in class, if $f(z) = e^{2z}$, then

$$f(z) = e^{2x}\cos(2y) + ie^{2x}\sin(2y) = u + iv.$$

Thus

$$f'(z) = u_x + iv_x = 2e^{2x}\cos(2y) + i2e^{2x}\sin(2y) = 2e^{2z}$$

just as we expected.

Problem 3. Use Proposition 3 to show the following are analytic.

(a) If $f(z) = \cos(z)$, then $f'(z) = \sin(z)$.
(b) If $f(z) = \sin(z)$, then $f'(z) = \cos(z)$.
(c) If $f(z) = \log(z)$, then $f'(z) = \dfrac{1}{z}$.

(In the case of $\sin(z)$ and $\cos(z)$ we have already seen they are analytic by anther method. But as a review of what we did last week, write out the real and complex parts of $\sin(z)$ and $\cos(z)$ and show they satisfy the CR equations and use Proposition 3 to give anther derivation that their derivatives are what they should be.)

While at this point is not clear there is much relationship between analytic function and functions that can be expressed as a convergent power series, it will turn out that the two are closely related. Here is a start on that

Proposition 4. *Consider the power series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Assume this converges for $z = z_1$. Then the series converges for all z with $|z| < |z_1|$.

Problem 4. Prove this. Before starting we make a few observations. As the series for $f(z_1)$ converges, the terms go to zero. That is $\lim_{n\to\infty} a_n z_1^n = 0$. This implies the terms are bounded, that is there is a constant, C, so that

$$|a_n z_1^n| \le C.$$

Define

$$r = \frac{|z|}{|z_1|} = \left| \frac{z}{z_1} \right|.$$

By hypothesis $|z| < |z_1|$, so

$$r < 1.$$

Thus by our basic results about geometric series

$$\sum_{n=0}^{\infty} Cr^n < \infty \tag{1}$$

Now proceed with the proof as follows.

(a) Show $|a_n z^n| \le Cr^n$. *Hint:* $|a_n z^n| = |a_n z_1^n||z/z_1|^n$.
(b) Finish the proof by use of the comparison theorem (look this up if you have forgotten it) part (a) and (1).

Problem 5 (Not to be handed in). Review the definition of the gradient and the chain rule for functions of two variables. In particular that the gradient of a function is orthogonal to the level curves of the function.