Mathematics 300

Quiz 32 Name: Key

You must show your work to get full credit.

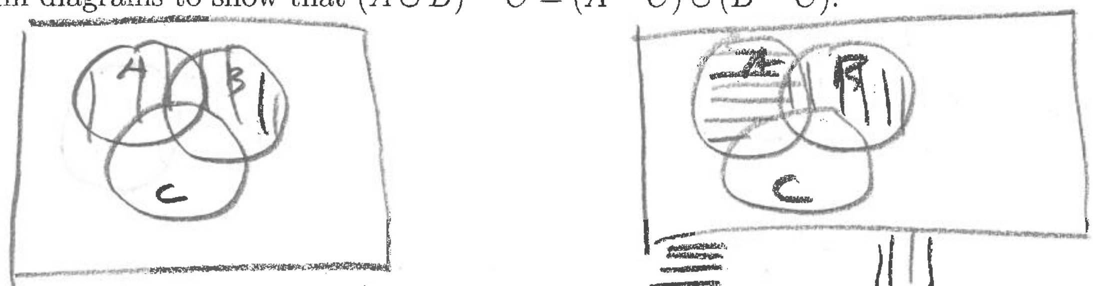
1. Prove or give a counter example: Every even integer is the some of two odd integers.

This is true. Let n be an even integer. Then $n = 2q$ for some integer q, and thus

$$n = 2q = 2q+1+(-1)$$

and $2q+1$ and (-1) are odd. Thus n is the sum of two odd numbers

2. Use Venn diagrams to show that $(A \cup B) - C = (A - C) \cup (B - C)$.



$||| = (A \cup B) - C$ $\equiv$ $A-C$ $|||$ $B-C$

The diagrams have same shaded area

3. What is the quotient and reminder when -32 is divided by 7?

Quotient is -5 Remainder is 3

$$-32 = -35 + 3$$
$$= -5(7) + 3$$

4. Find the sum $S = \sum_{k=0}^{10} 3(2)^k$. $= 3 + 3(2) + \cdots + 3(2)^{10}$ geometric series so

$$S = \frac{\text{first} - \text{next}}{1 - \text{ratio}}$$
$$= \frac{3 - 3(2)^{11}}{1-2} = 3(2)^{11} - 3 = 6141$$

5. Find the sum of the following twenty terms $31 + 32 + \cdots + 50$.

The sums is 810

This is arithmetic series so

$$\text{Sum} = (\text{\# of terms})(\text{Average})$$
$$= 20\left(\frac{31+50}{2}\right) = 10(81) = 810$$

6. What is wrong with the following statement of the division algorithm. "For unique integers a and b with $b > 0$ there are integers q and r with

$$a = qb + r \qquad \text{and} \qquad 0 \le r < b.\text{"}$$

The theorem does not hold for unique a and b, but for all a and b.

7. If chicken nuggets are only sold in boxes with two or three nuggets, show that it is possible to buy exactly n nuggets for any $n \ge 2$.

Base cases $2 = (2)$, $3 = (3)$, $4 = 2(2)$, $5 = (2)+(3)$
$6 = 2(3) = 3(2)$.

Induction step Assume that we can buy exactly k nuggets for $k \geq 6$.

Then to get $k+1$ nuggets either

Case 1 Return a box of 2 and replace it with a box of 3 to get $k-2+1 = k+1$

Case 2 There are no boxes of size 2. Then all boxes are of size 3. Return one of those and replace with a box of size 3 to get $k-3+2(2) = k+1$. done

8. Use induction to show that $10^n \equiv 1 \pmod 9$ for all integers $n \ge 1$.

Base case $n = 1$. Then $10^1 - 1 = 10 - 1 = 9$ which is divisible by 9. $10 \equiv 1 \bmod n$.

Induction step Assume $10^k \equiv 1 \bmod 9$.

Then multiply this by 10 to get

$$10 \cdot 10^k \equiv 10 \cdot 1 \mod 9$$
$$10^{k+1} \equiv 10 \mod 9$$
$$10^{k+1} \equiv 1 \mod 9$$

as $10 \equiv 1 \bmod 9$. done

9. Let $f(x) = (x+1)e^x$.

(a) Compute the first four derivatives of $f(x)$.

$f'(x) =$ $(x+2)\ e^x$ $\qquad$ $f''(x) =$ $(x+3)e^x$

$f'''(x) =$ $(x+4)e^x$ $\qquad$ $f^{(4)}(x) =$ $(x+5)e^x$

(b) Make a conjecture for a formula for the n-th derivative $f^{(n)}(x)$.

$f^{(n)}(x) =$ $(x+n+1)\ e^x.$

(c) Prove your conjecture.

The base case of $n=1$ holds as
$f'(x) = f^{(1)}(x) = (x+2)e^x = (x+1+1)e^x.$

Induction step Assume $f^{(k)}(x) = (x+k+1)e^x$

Then
$$f^{(k+1)}(x) = ((x+k+1)e^x)'$$
$$= (x+k+1)'e^x + (x+k+1)(e^x)'$$
$$= 1e^x + (x+k+1)e^x$$
$$= (1+x+k+1)e^x$$
$$= (x+(k+1)+1)e^x.$$ done

10. Let a_n be defined by

$$a_{n+1} = 3a_n - 8, \qquad a_0 = 6$$

(a) Find the following

$a_1 =$ 10 $\qquad$ $a_2 =$ 22 $\qquad$ $a_3 =$ $58.$

$a_1 = 3a_0 - 8$
$= 3(6) - 8$
$= 18 - 8$
$= 10$

$a_2 = 3(a_1) - 8$
$= 3(10) - 8$
$= 30 - 8$
$= 22$

$a_3 = 3(a_2) - 8$
$= 3(22) - 8$
$= 66 - 8$
$= 58$

(b) Prove $a_n = 2(3^n) + 4$ Base case: $k=0$ $a_0 = 10 = 2\cdot(3^0) + 4$
$= 2 + 4 = 6$

so this holds

Induction step: Assume $a_k = 2(3^k) + 4.$

Then
$$a_{k+1} = 3(a_k) - 8$$
$$= 3(2(3)^k + 4) - 8$$
$$= 2(3)^{k+1} + 12 - 8$$
$$= 3(3)^{k+1} + 4$$

done

11. (a) Define $a \equiv b \pmod{n}$. a, b, n are integers, $n \geq 1$ and

$n \mid (a-b)$

(b) Prove that if a and b have the same remainder when divided by n that $a \equiv b \pmod{n}$.

If a and b have the same remainder when divided by n this means that

$$a = q_1 n + r_1 \qquad 0 \leq r_1 < n$$
$$b = q_2 n + r_2 \qquad 0 \leq r_2 < n.$$

and $r_1 = r_2$. Then

$$\begin{aligned} a - b &= q_1 n + r_1 - (q_2 n + r_2) \\ &= (q_1 - q_2)n + (r_1 - r_2) \\ &= qn + 0 \\ &= qn \end{aligned}$$

where $r_1 - r_2 = 0$ as $r_1 = r_2$ and $q = q_1 - q_2$ is an integers. Thus $n \mid (a-b)$ and so

$a \equiv b \mod n$