Mathematics 300

Quiz 34 Name: Key

You must show your work to get full credit.

1. Let A be the set of all squares of odd integers and $B = \{x \in \mathbb{Z} : x \equiv 1 \pmod 4\}$.

(a) List four elements of A. $1^2=1,\ 3^2=9,\ 5^2=25,\ 7^2=49$

(b) List four elements of B. $1, 5, 9, 13$

(others are $-3, -7 \ldots$)

(c) Prove $A \subseteq B$.

Let $x \in A$. By the definition of A, x is the square of an odd numbers. So

$x = (2q+1)^2$ for some integer q.

Then

$x = 4q^2 + 4q + 1$

so $x - 1 = 4q^2 + 4q = 4(q^2+q)$.

so $4 \mid (x-1)$ (as $x-1$ is $4 \times q'$ where $q' = q^2 + 1 \in \mathbb{Z}$. Thus

$x \equiv 1 \pmod 4$.

(d) Prove $B \not\subseteq A$.

Let $5 \in B$ as $5 \equiv 1 \mod 4$. But 5 is not the square of any integers so $5 \notin A$. Thus B is not a subset of A.

2. Let

$$A = \{6u - 4v : u, v \in \mathbb{Z}\} \qquad B = \text{Set of even integers.}$$

(a) List four elements of A. $0, 6, 2, -2$

$6(0) - 4(0) = 0 \qquad 6(1) - 4(1) = 2$

$6(1) - 4(0) = 6 \qquad 6(3) - 4(5) = 18 - 20 = -2$

(b) List four elements of B. $-4, -2, 0, 2, 4, \ldots$

Show $A = B$. This is done in two steps:

(c) Prove $A \subseteq B$.

Let $x \in A$. Then $x = 6u - 4v$ for some $u, v \in \mathbb{Z}$.
So $x = 2(3u - 2v)$ and thus x is 2 times the integer $q = 3u - 2v$. So x is even. That is $x \in B$.
Thus every element of A is an element of B

(d) Prove $B \subseteq A$. Let $x \in B$. Then x is even by the definition of B. Therefore

$$x = 2q$$

for some integer q. Note $2 = 6 - 4$. So

$$\begin{aligned} x &= 2q \\ &= (6-4)q \\ &= 6q - 4q \\ &= 6u - 4v \end{aligned}$$

where $u = v = q$.
Thus $x \in A$. So $B \subseteq A$

(e) Write the punch line.

We have $A \subseteq B$ and $B \subseteq A$ so $A = B$.

3. Let $P = \{x \in \mathbb{R} : |x - 2| < 2\}$, and let S be the closed interval $S = [0, 5]$. Prove $P \subseteq S$.

(a) Solve the inequality $|x - 2| < 2$ and write P in interval notation.

$P =$ $(0, 4)$

We know

$|x-2| < 2$ is equivalent to

$-2 < x-2 < 2$

add 2 to this

$-2+2 < x-2+2 < 2+2$

$0 < x < 4$

i.e. $x \in (0, 4)$

(b) Show $P \subseteq S$.

If $x \in P = (0,4)$, then

$0 < x < 4$

so $0 \leq 0 < x < 4 \leq 5$

i.e. $0 \leq x \leq 5$.

Thus $x \in [0, 5]$.

This shows $P \subseteq S$.

4. The ***power set***, denoted by $\mathcal{P}(A)$ of a set A is the set whose elements are all the subsets of A.

(a) What is the power set of $\varnothing$?

$\mathcal{P}(\varnothing) =$ $\{\varnothing\}$

(b) What is the power set of $\{1, 2, 3\}$?

$\mathcal{P}(\{1, 2, 3\}) =$ $\{\varnothing, \{1\}, \{2\}, \{3\},$ $\{1,2\}, \{1,3\}, \{2,3\},$ $\{1,2,3\}\}$.

Or all on one line

$\{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

(c) What is $(\{1\}))$?

$\mathcal{P}(\mathcal{P}(\{1\})) =$

$\mathcal{P}(\{1\}) = \{\varnothing, \{1\}\}$

$\mathcal{P}(\mathcal{P}(\{1\})) = \{\varnothing, \{\varnothing\}, \{\{1\}\}, \{\varnothing, \{1\}\}\}$