Mathematics 300

Quiz 35 Name: Key

You must show your work to get full credit.

1. Let $A = \{x \in \mathbb{Z} : x \equiv 2 \pmod 5\}$ and $B = \{y \in \mathbb{Z} : y \equiv 7 \pmod{10}\}$.

(a) List at least 5 elements of A. $-8, -3, 2, 7, 12, 17, 22$

(b) List at least 5 elements of B. $-3, 7, 17, 27, 37, \ldots$

(c) Is $A \subseteq B$, prove or give a counterexample.

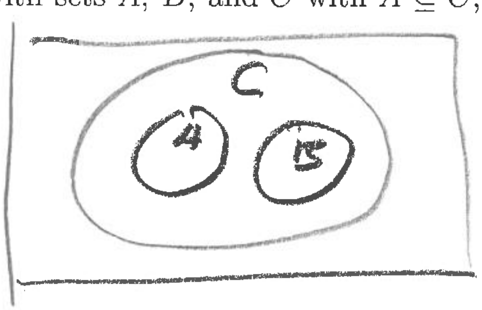
This is false. $2 \in A$, but $2 \notin B$ so $A \not\subseteq B$

(d) Is $B \subseteq A$, prove or give a counterexample.

True Let $y \in B$. Then $y \equiv 7 \mod 10$. That is $10 \mid (y-7)$ so $y - 7 = 10q$ for some $q \in \mathbb{Z}$.

Then
$$y = 10q + 7$$
$$= 10q + 5 + 2$$
$$= 5(2q+1) + 2$$
$$= 5q' + 2$$
where $q' = 2q + 1 \in \mathbb{Z}$. Then
$$y - 2 = 5q'$$
so $5 \mid y - 2$. That is $y \equiv 2 \mod 5$. Thus $y \in A$

2. Draw a Venn diagram with sets A, B, and C with $A \subseteq C$, $B \subseteq C$ and with A and B disjoint (that is $A \cap B = \emptyset$).



3. Prove the sets $S = \{x \in \mathbb{Z} : x \equiv 2 \pmod 7\}$ and $T = \{y \equiv 3 \pmod{14}\}$ are disjoint.

Towards a contradiction assume $S \cap T \neq \emptyset$. Then let $x \in S \cap T$.

Then $x \equiv 2 \bmod 7$ and $x \equiv 3 \bmod 14$

so

$7 \mid x-2$ and $14 \mid x-3$.

Then
$x = 7q_1 + 2$ $\quad x = 14q_2 + 3$

for some $q_1, q_2 \in \mathbb{Z}$.

set these equal.

$7q_1 + 2 = 14q_2 + 3$

$7q_1 - 14q_2 = 3 - 2$

$7(q_1 - 2q_2) = 1$

$7q = 1$

where $q = q_1 - 2q_2 \in \mathbb{Z}$. But solving gives $q = \frac{1}{7} \notin$ contradiction.

4. Prove: If k is an integer and $3 \mid k^2$, then $3 \mid k$.

We prove the contrapositive: If $3 \nmid k$, then $3 \nmid k^2$.

If $3 \nmid k$ then either $k \equiv 1 \bmod 3$ or $k \equiv 2 \bmod 3$.

<u>case 1</u> If $k \equiv 1 \bmod 3$, then
$k^2 \equiv 1^2 \equiv 1 \bmod 3$

so $3 \nmid k^2$.

<u>case 2</u> If $k \equiv 2 \bmod 3$, then
$k^2 \equiv 2^2 \equiv 4 \equiv 1 \bmod 3$.

so $3 \nmid k^2$ in this case.

<u>done</u>

5. Use Problem 4 to prove $\sqrt{3}$ is irrational.

Towards a contradiction assume $\sqrt{3}$ is rational say $\sqrt{3} = \frac{p}{q}$ with $p, q \in \mathbb{Z}$ and this fraction in lowest terms. Then squaring gives

$$3 = \frac{p^2}{q^2}$$

so $3q^2 = p^2$ $(*)$

This implies $3 \mid p^2$ and by problem 4 this implies $3 \mid p$. so $p = 3m$ for some $m \in \mathbb{Z}$. Use this in $(*)$ to get

$$3q^2 = (3m)^2 = 9m^2$$

$$q^2 = 3m^2$$

so $3 \mid q^2$. Thus $q = 3n$ for some n.

But then

$\frac{p}{q} = \frac{3m}{3n}$ is not in lowest terms contradiction

6. For integers x and y prove: If $3 \mid x$ and $2 \mid y$, then $12 \mid (5xy^2 + 10xy)$.

If $3 \mid x$ and $2 \mid y$, then $x = 3m$ and $y = 2n$ for some $m, n \in \mathbb{Z}$.

so

$$5xy^2 + 10xy = 5(3m)(2n)^2 + 10(3m)(2n)$$
$$= 60mn^2 + 60mn$$
$$= 12(5mn^2 + 5mn)$$
$$= 12q$$

where $q = 5mn^2 + 5mn \in \mathbb{Z}$. Thus $12 \mid 5xy^2 + 10xy$

done

7. What is the negation of the statement: "For all $x \in \mathbb{R}$ there is a $n \in \mathbb{N}$ with $n > x$".

there exists $x \in \mathbb{R}$ so that for all $n \in \mathbb{N}$, $n \leq x$.

8. (a) Define what it means for r to be a rational number.

$r = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$.

(b) Prove or give a counterexample: If a and b are irrational, then so is the product ab.

This is false. Let $a, b = \sqrt{2}$. $\sqrt{2}$ is irrational, but $ab = (\sqrt{2})^2 = 2$ is rational

(c) Prove or give a counterexample: If $a \neq 0$ is rational and b is irrational, then the product ab.

This is true. Towards a contradiction assume a is rational, b is irrational, but the product ab is rational. Then for some integers m, n, p, q with $m, n, p, q \neq 0$

$$a = \frac{m}{n}, \; ab = \frac{p}{q}$$

$$b = \frac{1}{a}\left(\frac{p}{q}\right) = \frac{n}{m}\left(\frac{p}{q}\right) = \frac{np}{mq} = \frac{p'}{q'}$$

where $p', q' \in \mathbb{Z}$. This implies b is rational, contradicting that it is irrational

done