Mathematics 300

Quiz 36 Name: Key

You must show your work to get full credit.

1. Prove if a and b are odd integers, then $(a-1)(b-1)$ is divisible by 4.

Assume a and b are odd. Then there are integers q_1, q_2 with

$$a = 2q_1+1, \quad b = 2q_2+1.$$

Then

$$(a-1)(b-1) = (2q_1+1-1)(2q_2+1-1) = 4q_1q_2 = 4q$$

where $q = q_1q_2$ is an integer. Thus $4 \mid (a-1)(b-1)$.

Recall at ***Pythagorean triple*** is a list of three natural numbers a, b, c with $a^2 + b^2 = c^2$

2. Find all Pythagorean triples of the form $m-1$, $m+1$, $m+3$.

If $a=m-1$, $b=m+1$, $c=m+3$ is a Pythagorean triple, then

$$a^2+b^2=c^2$$

i.e.

$$(m-1)^2+(m+1)^2=(m+3)^2$$

$$m^2-2m+1+m^2+2m+1 = m^2+6m+9$$

$$2m^2+2 = m^2+6m+9$$

$$m^2-6m-7=0$$

$$(m-7)(m+1)=0$$

So the only possible choices for m are $m=-1, 7$. But m must be positive thus $m=7$, and the triple is

$$a, b, c = m-1, m+1, m+3 = \underline{6, 8, 10}$$

3. Show that there is no Pythagorean triple, a, b, c where all three of a, b, and c are odd.

Towards a contradiction assume there are odd integers a, b, c with $a^2+b^2=c^2$.

Then, as squares of odd numbers are odd, a^2, b^2, c^2 are all odd. But then a^2+b^2 is the sum of two odd numbers and therefore is even. So

$$a^2+b^2=c^2$$

is impossible as a^2+b^2 is even and c^2 is odd.

4. Make a truth table for $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$. Is this a tautology?

P	Q	$P \to Q$	$\neg Q$	$\neg P$	$\neg Q \to \neg P$	$(P\to Q) \leftrightarrow (\neg Q \to \neg P)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Yes, it is a tautology

5. (a) What is the negation of the statement: $(\exists r \in \mathbb{Q})(r^3 = 2)$

(b) Write both $(\exists r \in \mathbb{Q})(r^3 = 2)$ and its negation as English sentences with no symbols.

(a) $(\forall r \in \mathbb{Q})(r^3 \neq 2)$

$(\exists r \in \mathbb{Q})(r^3=2)$: there exists a rational number whose cube is 2.

$(\forall r \in \mathbb{Q})(r^3 \neq 2)$: For all rational numbers the cube of the number is not 2.

6. Prove or give a counterexample: If a, b, c are integers and $a \mid bc$ then $a \mid b$ or $a \mid c$.

Counter example: $a=6$, $b=2$, $c=3$. Then $a=6 \mid 6 = b \cdot c$, but $6 \nmid 2$ and $6 \nmid 3$.

7. Prove or give a counterexample: If $a^2 \equiv 0 \pmod 9$, then $a \equiv 0 \pmod 9$.

counterexample: $a = 3$. Then $a^2 = 9 \equiv 0 \pmod 9$
but $a = 3 \not\equiv 0 \mod n$.

8. Prove or give a counterexample: If $a^2 \equiv 0 \pmod 3$, then $a \equiv 0 \pmod 3$.

We prove the contrapositive: If $a \not\equiv 0 \pmod 3$
Then $a^2 \not\equiv 0 \mod 3$. There are two cases:
$a \equiv 1 \mod 3$ and $a \equiv 2 \mod 3$

Case 1 $a \equiv 1 \pmod 3$. Then $a^2 = 1 \not\equiv 0 \mod 3$
So this case holds

Case 2 $a \equiv 2 \pmod 3$. Then $a^2 = 4 \equiv 1 \not\equiv 0 \mod 3$.
And so this case also holds.

9. Prove or give a counterexample: If n is odd, then $n^2 \equiv 1 \pmod 4$

Proof If n is odd, then $n = 2q+1$ for some integer q. Then

$$n^2 - 1 = (2q+1)^2 - 1$$
$$= 4q^2 + 4q + 1 - 1$$
$$= 4(q^2 + q)$$
$$= 4Q$$

where $Q = q^2 + q \in \mathbb{Z}$. Thus $4 \mid (n^2 - 1)$.
This is the definition of
$$n^2 \equiv 1 \mod 4$$

10. Prove that if $a^2 + a + 1$ is irrational, then so is a.

We prove the contrapositive. If a is rational, then so is a^2+a+1. If a is rational, then $a = \frac{p}{q}$ with $p, q \in \mathbb{Z}$. So

$$a^2 + a + 1 = \left(\frac{p}{q}\right)^2 + \left(\frac{p}{q}\right) + 1 = \frac{p^2 + pq + q^2}{q^2} = \frac{m}{n}$$

where $m = p^2 + pq + q^2$, $n = q^2$ are integers. Thus
$a^2 + a + 1 \in \mathbb{Q}$

11. We know that $\sqrt{3}$ is irrational. Use this to show $2\sqrt{3} - 4$ is irrational.

Towards a contradiction assume $2\sqrt{3}-4$ is rational. Then

$$2\sqrt{3} - 4 = \frac{p}{q}$$

where $p, q \in \mathbb{Z}$. Solve for $\sqrt{3}$

$$\sqrt{3} = \frac{1}{2}\left(\frac{p}{q} + 4\right) = \frac{1}{2}\left(\frac{p+4q}{q}\right) = \frac{p+4q}{2q} = \frac{m}{n}$$

where $m = p + 4q$, $n = 2q$ are integers.
This implies $\sqrt{3}$ is rational, a contradiction.

12. Prove: If a, b, c are integers with $a + b + c$ even, then for every integers n the number $an^3 + bn^2 + cn + 1$ is odd. There are two cases n is even (i.e. $n \equiv 0 \mod 2$) or n is odd (i.e. $n \equiv 1 \mod 2$)

Case 1 $n \equiv 0 \mod 2$ Then

$$an^3 + bn^2 + cn + 1 \equiv 0 + 0 + 0 + 1$$
$$\equiv 1 \pmod 2$$

so $an^3 + bn^2 + cn + 1$ is odd in this case

Case 2 $n \equiv 1 \mod 2$. Then

$$an^3 + bn^2 + cn + 1 \equiv a \cdot 1^3 + b 1^2 + c \cdot 1 + 1 \pmod 2$$
$$\equiv (a + b + c) + 1 \pmod 2$$
$$\equiv 0 + 1 \equiv 1 \pmod 2$$

as $a + b + c$ is even so $a + b + c \equiv 0 \mod 2$

Thus $an^3 + bn^2 + cn + 1$ is odd in this case also

13. It is true that for any integer a if $3 \mid a^3$, then $3 \mid a$. Use this to prove $\sqrt[3]{3}$ is irrational.

Towards a contradiction assume $\sqrt[3]{3}$ is rational with $\sqrt[3]{3} = \frac{p}{q}$ in lowest terms. Then $q\sqrt[3]{3} = p$ so $3q^3 = p^3$. This implies $3 \mid p^3$ so then $3 \mid p$. Then $p = 3a$ for some $a \in \mathbb{Z}$. Use this in $3q^3 = p^3$ to get $3q^3 = (3a)^3 = 27a^3$ i.e. $q^3 = 9a^3 = 3(3a^3)$. Thus $3 \mid q^3$, which implies $q = 3b$ for some $b \in \mathbb{Z}$. Then $\frac{p}{q} = \frac{3a}{3b}$ is not in lowest terms, a contradiction.

14. Find the sum $1 + 2 + \cdots + 50$. $=$ (number of terms) (average)

$= 50\left(\frac{1+50}{2}\right)$

$= 25(51)$ ⟵ ok to stop here

$= 1275$

15. Find the sum $5 - 5(3) + 5(3)^2 - 5(3)^3 + 5(3)^4 - 5(3)^5$. $= \frac{\text{first} - \text{next}}{1 - \text{ratio}}$

$= \frac{5 - 5(3)^6}{1-(-3)} = \frac{5 - 5(3)^6}{4}$ ⟵ ok to stop here

$= -910$

16. Let a sequence $a_1, a_2, a_3, \ldots$ be defined by

$a_1 = 1,$ ~~[illegible]~~. $a_n = \sqrt{10 + a_{n-1}}$

Prove $a_n < 6$ for all n.

We use induction:

Base case: $n = 1$. Then $a_1 = 1 < 6$.

Induction hypothesis: $a_k < 6$.

Induction goal: $a_{k+1} < 6$.

Induction step: $a_{k+1} = \sqrt{10 + a_k}$

$< \sqrt{10 + 6}$ (as $a_k < 6$)

$= 4 < 6$. done

17. Use induction to prove if $a \equiv b \pmod m$, then for all natural numbers n we have $a^n \equiv b^n$. (You may assume we know that if $a \equiv b \pmod m$ and $c \equiv d \pmod m$, then $ac \equiv bd \pmod m$).

Base case: $a^1 \equiv b^1 \bmod m$ holds as $a \equiv b \pmod m$.

Induction hypothesis: $a^k \equiv b^k \pmod m$

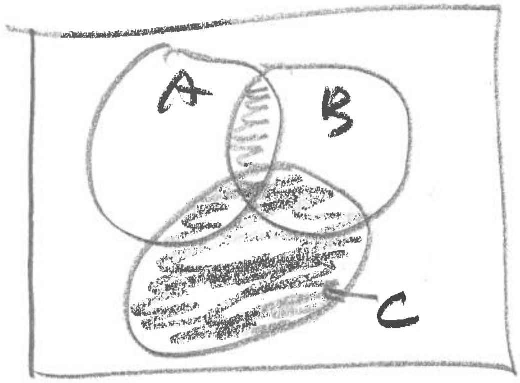
Induction goal: $a^{k+1} \equiv b^{k+1} \pmod m$

Induction step: let $c = a^k$, $d = b^k$, then

$a^{k+1} = ac \equiv bd = b^{k+1} \pmod m$

done

18. Draw the Venn diagram for $(A \cap B) \cup C$.



19. Let a be a constant and let $f(x) = xe^{ax}$. Prove that for all positive integers the n-th derivative of f is

$$f^{(n)}(x) = (a^n x + na^{n-1})e^{ax}.$$

We use induction:

Base case: $n=1$. $f^{(1)}(x) = f'(x) = (xe^{ax})'$

$= x(e^{ax})' + x'(e^{ax})$

$= x(ae^{ax}) + 1e^{ax}$

$= (ax+1)e^{ax}$

$= (a^1 x + 1 \cdot a^0)e^{ax}$

so this case holds

Induction hypothesis: $f^{(n)}(x) = (a^n x + na^{n-1})e^{ax}$

Induction goal: $f^{(n+1)}(x) = (a^{n+1}x + (n+1)a^n)e^{ax}$

Induction step:

$f^{(n+1)}(x) = (f^{(n)}(x))' = ((a^n x + na^{n-1})e^{ax})'$

$= (a^n x + na^{n-1})ae^{ax} + (a^n + 0)e^{ax}$

$= ((a^{n+1}x + na^n) + a^n)e^{ax}$

$= (a^{n+1}x + (n+1)a^n)e^{ax}.$ done