

Mathematics 554 Homework.

I decided you should all know the basics about at least some of the conic sections. One definition of an ellipse is the set of points $P = (x, y)$ such that there are two points P_1 and P_2 such that the distance of P to P_1 and the distance of P to P_2 is a constant. It is convenient to call this distance $2a$.

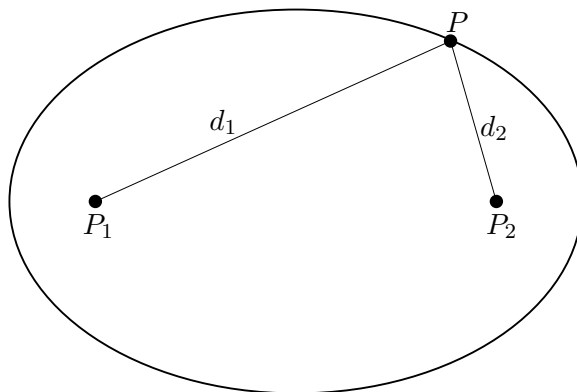


FIGURE 1. An ellipse is defined by two points P_1 and P_2 , the foci, and a distance $2a$ and is the set of points P such that the sum of the distances $d_1 := \|P - P_1\|$ and $d_2 := \|P - P_2\|$ is $d_1 + d_2 = 2a$.

To get an equation for this we let $\|P_2 - P_1\| = 2c$ be the distance between P_1 and P_2 . We use a coordinate system where the coordinate of P_1 is $(-c, 0)$ and the coordinate of P_2 is $(c, 0)$. Let $P = (x, y)$. Then the picture becomes

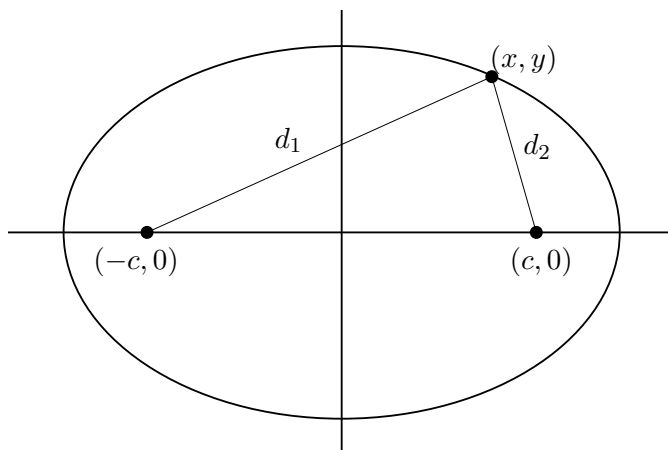


FIGURE 2. We have the coordinates so that the origin is on the midpoint between the foci and the foci are at $(-c, 0)$ and $(c, 0)$.

With this set up

$$d_1 = \sqrt{(x+c)^2 + y^2}, \quad d_2 = \sqrt{(x-c)^2 + y^2}$$

Therefore the equation, $d_1 + d_2 = 2a$, for the ellipse becomes

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

This is ugly, so let's try to give it a makeover that will get rid of the square roots. Rewrite as

$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

Squaring this gives

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

Problem 1. Show this can be rewritten as

$$a^2 - cx = a\sqrt{(x-c)^2 + y^2}$$

An improvement, but still far from pretty. But we can square this to get

$$(a^2 - cx)^2 = a^2((x-c)^2 + y^2)$$

Problem 2. Show this can be rewritten as

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

Then divide by $a^2(a^2 - c^2)$ to get

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

To simplify notation it is traditional to set $b = \sqrt{a^2 - c^2}$. Then the equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

In terms of a , b , and c the picture is

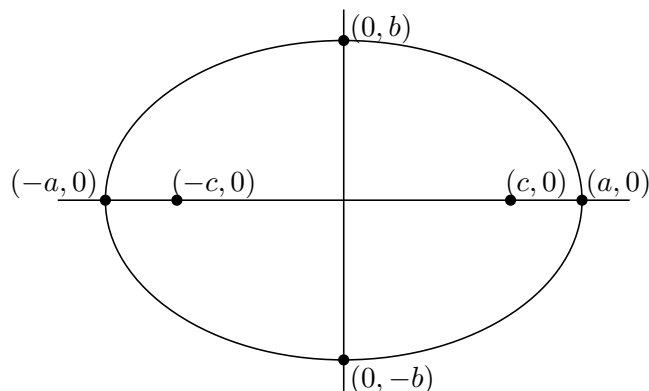


FIGURE 3. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is as shown. It is easy to read a and b off the graph. Then $c = \sqrt{a^2 - b^2}$

Here is some practice on line integrals.

Problem 3. Parameterize the right half of the ellipse given by

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

and use your parameterization to compute

$$\int_{\mathcal{C}} -y \, dx + 2x \, dy.$$

Hint: This is a problem where using a computer system to do the integrals is reasonable.

Problem 4. Let \mathcal{C} be the line segment from $(1, 3)$ to $(-1, 2)$. Compute

$$\int_{\mathcal{C}} y^2 \, dx + 2xy \, dy$$

Problem 5. Compute the length of the parabola $y = x^2$ between $x = -1$ and $x = 1$. *Hint:* While the integral involved is not outrageous, it is messy enough that using a computer to do it is reasonable.

Problem 6. Let \mathcal{C} be the curve parameterized by

$$\mathbf{r}(t) = \left(\frac{t^2}{2}, \frac{t^3}{3} \right)$$

for $0 \leq t \leq 1$. Set up the integral for

$$\int_{\mathcal{C}} 6xy \, ds$$

where s is arclength along \mathcal{C} .

Problem 7. Use Green's Theorem to evaluate the integral

$$\oint_D (xy + e^{x^2}) dx + (x + \cos(y^3)) dy$$

where D is the square defined by $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.

Problem 8. Let \mathcal{C} be the curve in \mathbb{R}^3 given by

$$\mathbf{r}(t) = \langle a \cos(t), a \sin(t), bt \rangle$$

where $a, b > 0$. Such a curve is a **helix**. See figure 4 for an example and <https://www.desmos.com/3d/azkrnw1bqz> for an interactive version of this graph.

- (a) Find the length of this curve for $0 \leq t \leq 2\pi$.
 (b) Let \mathcal{S} be the part of this helix with $0 \leq t \leq 2\pi$. Compute the line integral

$$\int_{\mathcal{S}} -yz dx + xz dy + xy dz.$$

- (c) Find the tangent line to this helix at the point where $t = 2\pi$. Where does this tangent line intersect the x, y plane.

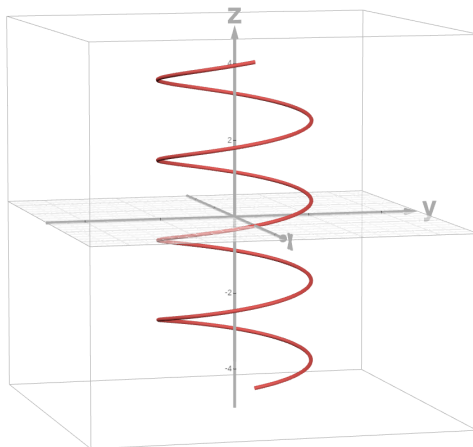


FIGURE 4. The helix $\mathbf{r}(t) = \langle a \cos(t), a \sin(t), bt \rangle$ with $a = 2$ and $b = 1/3$ and $-4\pi \leq t \leq 4\pi$.

Problem 9. (a) Show that the two curves

$$\mathbf{p}(t) = (t, t^2, t^3) \quad \text{and} \quad \mathbf{q}(t) = (t + 1, 4t^2, t + 7)$$

intersect and find their point of intersection.

(b) Show the two curves

$$\mathbf{u}(t) = (1 + t^2, t^3, e^t), \quad \text{and} \quad \mathbf{v}(t) = (-t^2, 1 + t, -3t^4)$$

never intersect.

Problem 10. Let \mathcal{H} (H for hyperbola) be the set of points (x, y) so that

$$\text{Distance}((x, y), (-c, 0)) - \text{Distance}((x, y), (c, a)) = 2a$$

for constants $a, c > 0$. This is equivalent to

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a.$$

Do a calculation like that the one we did for the ellipse to get a form of this equation that has no square roots.