Mathematics 554 Homework.

In this homework we will practice setting up line and surface integrals. In some cases I will want the integrals evaluated, but, as I have said in class repeatedly, with modern technology the skill of being able to do integrals by hand is not nearly as important as it use to be.

Let us start with some line integrals.

Problem 1. Let $\mathcal{C}$ be the curve parameterized by $\mathbf{r}\colon [0, c] \to \mathbb{R}^3$ be a curve in $\mathbb{R}^3$ given by

$$\mathbf{r}(t) = (a\cos(t), a\sin(t), bt)$$

where $a, b, c > 0$ are constants. (See Figure 1.)

(a) Compute the length of this curve. (This is a case where it is easy to compute the integral involved.)

(b) Compute the integral

$$\int_{\mathcal{C}} -y\,dx + x\,dy + z^3\,dz$$

(This is also one where the integral is reasonable.)

(c) Set up the integral for

$$\int_{\mathcal{C}} \sqrt{x^2 + y^2 + z^2}\,ds$$

where s is arclength along $\mathcal{C}$. (This integral is doable, but is a bit of a mess. Wolfram alpha refused to do it.)

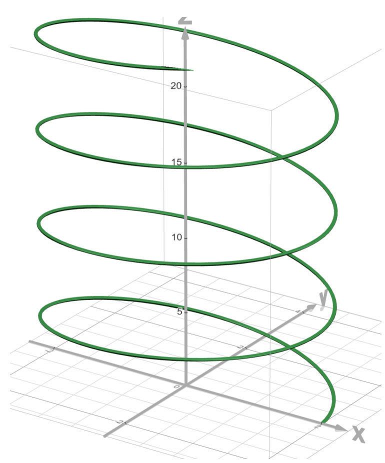

Figure 1. The helix of Problem 1 with $a = 2$, $b = 1/5$, and $c = 8\pi$. The interactive version is at: `https://www.desmos.com/3d/1vkinv2rv0`

Definition 1. Let $\mathcal{C}$ be a curve and f a function defined on $\mathcal{C}$. Then the ***average value*** of f on $\mathcal{C}$ is

$$\text{Average of } f \text{ on } \mathcal{C} = \frac{1}{L}\int_{\mathcal{C}} f\,ds$$

where s is arclength along $\mathcal{C}$. □

Problem 2. Let $\mathcal{C}$ be the ***twisted cubic*** parameterized by

$$\mathbf{r}(t) = (t, t^2, t^3)$$

for $t \in \mathbb{R}$. This is curve with a lot of history see:
`https://mathcs.holycross.edu/~little/TwistedMonthly.pdf`

(a) Let $\mathcal{T}$ be the part of the twisted cubic with $0 \le t \le 2$. Compute

$$\int_{\mathcal{T}} z\,dx + y^2\,dy + x^3\,dz.$$

(This integral can be done by hand.)

(b) Set up the integral for the length of $\mathcal{T}$ and evaluate in numerically. (This one is impossible by hand.)

(c) Compute the average distance of a point of $\mathcal{T}$ from the origin. That is if L is the length of $\mathcal{T}$ compute

$$\frac{1}{L}\int_{\mathcal{T}} \sqrt{x^2+y^2+z^2}\,ds$$

Again set up the integral and then compute it numerically.

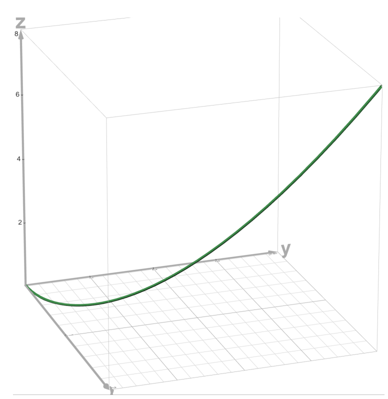

FIGURE 2. The twisted cubic $\mathcal{T}$. The interactive version is at:
`https://www.desmos.com/3d/bln4crz2k7`

On to surface integrals. Let $\mathcal{S}$ be a surface in $\mathbb{R}^3$. We assume that we can parameterize this by $\mathbf{r}\colon U \to \mathcal{S}$ where U is an subset of $\mathbb{R}^2$. This will look like

$$\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v)).$$

We have seen that the area of $\mathcal{S}$ is

$$\operatorname{Area}(\mathcal{S}) = \iint_U \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA = \iint_U \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| du\, dv$$

and if f is a function on $\mathcal{S}$, then

$$\iint_{\mathcal{S}} f\, d\sigma = \iint_U f(\mathbf{r}(u,v)) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA.$$

In this equation $d\sigma$ is the surface area measure on $\mathcal{S}$ and $dA = du\, dv = dv\, du$ is the area measure on U.

Definition 2. Let $\mathcal{S}$ be a surface and $\mathbf{r}\colon U \to \mathcal{S}$ a parameterization of $\mathcal{S}$. Then the ***unit normal*** to $\mathcal{S}$ is

$$\mathbf{n}(u,v) = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}$$

where we are using the subscript notation for partial derivatives

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} \qquad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}.$$

□

Definition 3. If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ then the ***flux*** of $\mathbf{F}$ through $\mathcal{S}$ is defined to be

$$\text{flux through } \mathcal{S} = \iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

□

We have seen that in terms of the parameterization this reduces to

$$\text{flux through } \mathcal{S} = \iint_U (\mathbf{r}_u \times \mathbf{r}_v) \cdot \mathbf{n}\, dA$$

Proposition 4. *Let U be a subset of $\mathbb{R}^2$ and $f\colon U \to \mathbb{R}$ a function on U. Then the graph, $z = f(x,y)$ is parameterized by*

$$\mathbf{r}(u,v) = (u, v, f(u,v)).$$

For this we have seen that

$$\begin{aligned}
\mathbf{r}_u &= \langle 1, 0, f_u \rangle \\
\mathbf{r}_v &= \langle 0, 1, f_v \rangle \\
\mathbf{r}_u \times \mathbf{r}_v &= \langle -f_u, -f_v, 1 \rangle \\
\|\mathbf{r}_u \times \mathbf{r}_v\| &= \sqrt{1 + f_u^2 + f_v^2} \\
d\sigma &= \|\mathbf{r}_u \times \mathbf{r}_v\| du\, dv = \sqrt{1 + f_u^2 + f_v^2}\, du\, dv \\
\mathbf{n} &= \frac{\langle -f_u, -f_v, 1 \rangle}{\sqrt{1 + f_u^2 + f_v^2}}.
\end{aligned}$$

□

Problem 3. Let $\mathcal{S}$ be the graph of $z = xy$ over the square $-1 \le u, v \le 1$. See Figure 3. Compute the following. (It is ok to give the answers numerically, but show your work for setting up the integrals.)

(a) The area of $\mathcal{S}$.
(b) The unit normal to $\mathcal{S}$.
(c) The integral
$$\iint_{\mathcal{S}} f \, d\sigma$$
where $f = x^2 + 2y^2 - z^2$.
(d) The flux of the vector field
$$\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + 5\mathbf{k}$$
through $\mathcal{S}$.

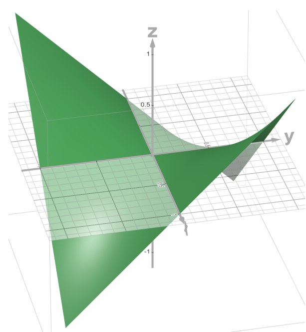

FIGURE 3. The graph of $z = xy$ over the square $-1 \le x, y \le 1$. The interactive version as at: `https://www.desmos.com/3d/irofrw7zox`

Proposition 5. *Let* $\mathbf{c}(u) = (x(u), y(u))$ *with* $a \le u \le b$ *be a curve in the place with* $x(t) > 0$. *Let us rotate this about the* y*-axis. The result can be parameterized by*
$$\begin{aligned}\mathbf{r}(u, v) &= x(u)(\cos(v), \sin(v), 0) + y(u)(0, 0, 1)\\ &= (x(u)\cos(v), x(u)\sin(v), y(u))\end{aligned}$$
with $a \le u \le b$ *and* $0 \le v \le 2\pi$. *Then*

(a) The first derivatives of $\mathbf{r}$ *are give by*
$$\begin{aligned}\mathbf{r}_u &= \langle x'(u)\cos(v), x'(u)\sin(v), y'(u)\rangle\\ \mathbf{r}_v &= \langle -x(u)\sin(v), x\cos(v), 0\rangle\end{aligned}$$
(b) $\mathbf{r}_u \times \mathbf{r}_v = \langle -x(u)y'(u)\cos(v), x(u)y'(u)\sin(v), x(u)x'(u)\rangle$.
(c) $\|\mathbf{r}_u \times \mathbf{r}_v\| = x(u)\sqrt{x'(u)^2 + y'(u)^2}$.

(d) The area of $\mathcal{S}$ is

$$\text{Area}(\mathcal{S}) = 2\pi \int_a^b x(u)\sqrt{x'(u)^2 + y'(u)^2}\, du$$

(e) If f is a function on $\mathcal{S}$, then

$$\iint_{\mathcal{S}} f\, d\sigma = \iint_U f(\mathbf{r}(u,v))x(u)\sqrt{x'(u)^2 + y'(u)^2}\, du\, dv.$$

Problem 4. Prove this this proposition. □

Problem 5. Let put Proposition 5 to work. Let $0 < a < b$ be constants. The circle of radius a with center $(b, 0)$ is parameterized by

$$\mathbf{c}(u) = (x(u), y(u)) = (b + a\cos(u), a\sin(u))$$

with $0 \le u \le 2\pi$. Revolving this around the y-axis is parametrized by

$$\begin{aligned}\mathbf{r}(u,v) &= (x(u)\cos(v), x(u)\sin(v), y(u))\\ &=((b + a\cos(u))\cos(v), (b + a\cos(u))\sin(v), a\sin(u))\end{aligned}$$

The resulting surface $\mathcal{S}$ is a ***torus***. See Figure 4

(a) Compute the area of $\mathcal{S}$.
(b) Set up the integral for computing

$$\iint_{\mathcal{S}} xyz^2\, d\sigma$$

and compute the value of this integral when $a = 1$ and $b = 2$.
(c) Let $\mathbf{F} = \langle -y, x, z^2 \rangle$. Set up the integral for the flux of $\mathbf{F}$ through $\mathcal{F}$ and compute the integral for when $a = 1$ and $b = 2$.

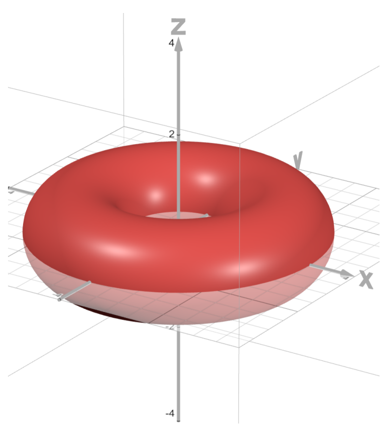

FIGURE 4. The torus of Problem 5 with $a = 1$ and $b = 2$. The interactive version is at: `https://www.desmos.com/3d/yjjzsrwbaw`